NON-NEGATIVE SYMMETRIC POLYNOMIALS AND ENTANGLED BOSONS

A Dissertation
Presented to
The Academic Faculty

By

Bharath. H. M

In Partial Fulfillment
of the Requirements for the Degree
Master of Science in the
School of Mathematics

Georgia Institute of Technology

June 2018

NON-NEGATIVE SYMMETRIC POLYNOMIALS AND ENTANGLED BOSONS

Approved by:

Prof. Grigoriy Blekherman, Advisor
School of Mathematics
Georgia Institute of Technology

Prof. Brian Kennedy
School of Physics
Georgia Institute of Technology

Prof. Sung Ha Kang
School of Mathematics
Georgia Institute of Technology

Date Approved: ,

The bringing together of theory and practice leads to the most favorable results; not only does practice benefit, but the sciences themselves develop under the influence of practice, which reveals new subjects for investigation and new aspects of familiar subjects

Pafnuty Chebyshev

To my parents, H. N. Madhusudhana and H. M. Bharathi.

ACKNOWLEDGEMENTS

An evening in August 2016, I came to a sudden realization that the elements of the density matrix of a separable quantum state (i.e., an un-entangled state) are moments of some underlying probability distribution function. This immediately connects the physicists' problem of telling which density matrices represent entangled states to the mathematicians' problem of telling which sets of numbers are legitimate moments of some probability distribution function. The latter, I knew, belongs to a class of problems called the *truncted-moment problem.* Following a quick literature search on truncated-moment problems, while looking for any progress on the specific problem that would correspond to entanglement between a pair of two level atoms, I discovered a recent paper [1] which considered the relevant problem, but the author was completely unaware of its connection to the entanglement problem in physics. Almost by an accident, I noticed that the author, Prof. Greg Blekherman, lived minutes away from my laboratory.

I met Prof. Greg the following week and expressed my interest in this subject and explained to him that it is related to a problem in quantum entanglement — something that I would be interested in pursuing at our Bose-Einstein condensation laboratory. Under Prof. Greg's guidance, I started working on the specific truncated moment problem, that relates to entanglement of a Bosonic many-body system. This was a side project to my Ph.D thesis work at Prof. Michael Chapman's laboratory.

Soon, I decided this problem is going to be my Math Masters thesis problem, and I would work on it alongside my Physics Ph.D. Prof. Greg happily agreed to this plan. I am grateful to him for his time and guidance. His insights have proved helpful in pulling me out of "dead ends" that I encountered while solving this problem.

When I expressed my strange idea of doing a Masters with a thesis in mathematics alongside a Ph.D in experimental physics to my advisor Prof. Michael Chapman, I was met with a rather surprising, encouraging response. Prof. Chapman said that I should

certainly explore my interests in mathematics alongside my work at his laboratory and that he would be happy to support me in this endeavor. I am grateful to Prof. Chapman for his enduring support and the infinite academic freedom that he has given me through my time as a Ph.D student at his laboratory.

The graduate coordinators of the school of Physics, Prof. Andrew Zangwill and Prof. David Ballantyne and the graduate coordinator of the school of Mathematics, Prof. Mohammad Ghomi have also been very supportive and cooperative in arranging for my double major.

Members of our laboratory, Matthew Boguslawski, Maryrose Barrios and Lin Xin have been extremely supportive to me throughout my time at the lab. I am thankful to all of them for being accommodating, making it possible for me to balance my double major. I am also thankful to my thesis committee members, Prof. Brian Kennedy and Prof. Sung Ha Kang for taking the time to review this work and proving their valuable feedback. I am grateful my parents, H. N. Madhusudhana and H. M. Bharathi and my sister, H. M. Madhuri for their endless support to my endeavors in Physics and Mathematics.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SUMMARY AND THESIS CONTRIBUTION

This thesis considers a problem in real algebraic geometry that is born out of many-body physics. The latter is the study of the physical properties of a system consisting of many particles, usually identical atoms. There is a natural, but not well appreciated connection between many-body physics and polynomials. For instance, the quantum state of a many-body system of atoms is represented by a multi-variate polynomial of degree equal to the number of atoms and the variables in the polynomial are the so-called *creation operators*. The number of variables, in the polynomial is the number of states of existence of each atom in the many-body system. Multiplication of two such polynomials corresponds to merging of the two many-body states represented by the two polynomials into a larger many-body system. The product of the two polynomials is the quantum state of the merged many-body system. Experimentally, this process is called *adiabatic merging*. Factoring a polynomial into two polynomials corresponds to splitting a many-body system into two smaller many-body systems, also known as a *mode split*. The study of multi-variate polynomial factorization, in particular, addressing questions such as which polynomials are factorizable would help us answer questions such as which many-body quantum states can be prepared by an adiabatic merger.

In a second thread of connection between polynomials and many-body physics, global minima of polynomial functions over a compact set correspond to the so-called ground states of many-body systems. The energy functional of a many-body system, also known as *mean-field energy*, is usually a polynomial with a degree equal to the degree of interaction i.e., quadratic terms correspond to two-body interaction, cubic terms correspond to three body interaction etc. The variables in the polynomial are the degrees of freedom of one atom and their domain is a compact set. The stable states of such a system are obtained by minimizing the energy functional over the range of the variables, which is a compact set.

In a third thread of connection between many-body physics and polynomials, the pro-

blem of determining whether a many-body state is *entangled* can be rephrased as a *truncated K-moment problem* (TKMP). We provide a brief explanation of the two italicized technical terms that appeared in the previous sentence.

We begin with the TKMP. The *moment problem* is the well-known problem of determining a probability distribution, i.e., a measure, starting from its moments. The *truncated moment problem* is the problem of determining a measure starting from a truncated subset of its moments. If the measure is defined over a compact set K, then the corresponding truncated moment problem is called *truncated K-moment problem*.

A TKMP is closely related to non-negativity of polynomials over the compact set K. A moment is the expectation value of a monomial under a measure. In convex analysis, the bipolar theorem asserts that the closure of the cone of moments of measures over K is the dual of the cone of polynomials containing the relevant monomials, that are non-negative over K. Therefore, characterizing the moment cone is equivalent to characterizing the cone of non-negative polynomials.

We next provide a brief discussion of entanglement and how it is related to TKMPs.

In a general setting, every physically observable parameter of a physical system is modeled by a random variable. The experimentally measured value of the parameter would be the first moment of the random variable. The set of of physical parameters of a many-body system are modeled by a set of random variables, together with a joint measure.

Based on the nature of the joint measure we may identify two classes of quantum states of a many-body system. If the joint measure can be factored into a product, then we say that the constituents of the many-body system are uncorrelated. If the measure can not be factored, we say that the constituents of the many-body system are *classically* correlated. The reason behind the unexpected appearance of the prefix "classical" will be clear soon.

Quite intriguingly, there are some states of a many-body system that do not belong to either of the classes mentioned above. The measured values of the parameters of such states can not be described as moments of a measure. There can be no measure — factorizable or

otherwise — whose moments are equal to the measured values of the parameters. They can be described only by a signed measure. Accordingly, there states are perceived to possess a higher level of correlations and are called *quantum correlated* states or *entangled* states.

One of the central problems in quantum entanglement is to show, using experimentally measured values of a set of parameters, that the underlying quantum state is entangled. Quite clearly, this problem is a TKMP — the objective is to determine whether the parameters measured in the laboratory are moments of some measure. In this thesis we consider an example of a TKMP that comes from a specific problem concerning entanglement in a system consisting of N number of identical *Bosons*. The latter is a term used to describe atoms that possess an exchange symmetry. That is, all physical properties of a system of N identical Bosons are invariant under the action of the symmetric group S_N. Therefore, the corresponding moment cone would be dual to the cone of non-negative symmetric polynomials.

This thesis is organized as follows. Chapter 1 provides an introduction to TKMP and non-negative polynomials. Chapter 2 starts with a brief summary of quantum entanglement and describes with examples how the problem of deciding whether a state is entangled or not is a TKMP. Previous work in the fields on many-body entanglement and in non-negative symmetric polynomials are summarized in Chapter 3. In Chapter 4, we summarize the statement of the problem considered in this thesis. In Chapter 5, we discuss our results on the exact characterization of the cone of symmetric polynomials that are non-negative over the compact set relevant to a many-body system of Bosons. In Chapter 6, we discuss our results on asymptotic characterization of the same cone, leading to an asymptotically tight entanglement criteria, for large N. We also compare this criterion with previous work on many-body entanglement. Finally, we discuss possible generalizations of the problem and possible experimental application of the results in the thesis.

<center>**CHAPTER 1**</center>

<center>**TKMP AND NON-NEGATIVE POLYNOMIALS**</center>

This chapter is a basic introduction to TKMP and is aimed at the non-expert audience who might be drawn to this thesis because of its connection to many-body physics. We begin with basic notations and definitions and briefly discuss the Bipolar theorem and explain how TKMP is related to non-negative polynomials.

1.1 Introduction

Let $\mathbf{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$. Let $\mathcal{N} = (n_1, n_2, \cdots, n_k) \in \mathbb{N}^k$ be a list of positive integers. We denote the monomial $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ by $\mathbf{x}^{\mathcal{N}}$. The variables $x_1, x_2, \cdots, x_k$ take real values and we denote their domain by $K \subseteq \mathbb{R}^k$. Let μ be a measure defined over K. Corresponding to every monomial of the form $\mathbf{x}^{\mathcal{N}}$, we may define a moment:

$$m_{\mathcal{N}} = \int_K \mathbf{x}^{\mathcal{N}} d\mu \tag{1.1}$$

The truncated K-moment problem is concerned with the following question: given $K \subset \mathbb{R}^k$, a collection of indicies $I = \{\mathcal{N}_1, \mathcal{N}_2, \cdots\} \subseteq \mathbb{N}^k$, and a corresponding collection of real numbers, $\{\gamma_1, \gamma_2, \cdots\}$, is there a measure μ defined over K such that

$$\gamma_i = m_i = \int_K \mathbf{x}^{\mathcal{N}_i}. \tag{1.2}$$

In other words, can the numbers a_i be recovered as the moments of some measure supported on K? In general, K is assumed to be a compact set. However, in this chapter, we do not impose such a restriction for the purpose of discussing a few general results. In the extreme case when $K = \mathbb{R}^k$ and $I = \mathbb{N}^k$, i.e., when all the moments are known, this

<center>1</center>

problem reduces the the classic moment problem. The latter is the problem of recovering a probability distribution (i.e., a measure) starting from the set of all of its moments. Indeed, not every set of numbers represent moments of a legitimate measure. For instance, when $k = 1$, the second moment is bounded below by the square of the first moment, i.e.,

$$\left(\int x d\mu \right)^2 \leq \int d\mu \int x^2 d\mu. \tag{1.3}$$

This follows from the Cauchy-Schwartz inequality. Therefore, this inequality is a necessary condition for a set of numbers to be moments. In general, a set of numbers have to satisfy multiple inequality criteria in order for them to represent moments of a measure. When $K = \mathbb{R}$, and $I = \mathbb{N}$, the resulting moment problem is known as the *Hamburger moment problem*. When $K = [0, \infty)$ and $I = \mathbb{N}$, the resulting moment problem is known as the *Stieltjes moment problem*. When $K = [0, 1]$ and $I = \mathbb{N}$ the resulting moment problem is called *Hausdorff moment problem*. All of the above three examples have complete solutions [2]. However, the problem is more non-trivial when $I \subset \mathbb{N}^k$, i.e., only a subset of the moments are known, in which case the problem is called a truncated moment problem [3].

If $K \subset \mathbb{R}^k$, it makes the problem much more non-trivial. A better formulation of this problem can be constructed using the notion of a moment cone.

1.2 The moment cone

Henceforth, we restrict ourselves to the case where $|I| = n < \infty$, which enables us to treat the set of the moments as components of a vector, $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{R}^n$. Similarly, the set of the relevant moments of a given measure can also be treated as the components of a vector, $\mathbf{m} = (m_1, m_2, \cdots, m_n)$, which we may refer to as the "moment vector". The TKMP would then be concerned with the question of whether this vector γ is in the set of

moment vectors. The latter, defined as

$$C = \left\{ \mathbf{m} : \quad \mathbf{m} = (m_1, m_2, \cdots, m_n) \in \mathbb{R}^n, \ m_i = \int_K \mathbf{x}^{\mathcal{N}_i} d\mu, \ \text{for some measure } \mu \right\}.$$
(1.4)

This set C is known as the moment cone corresponding to I and K. The TKMP can be phrased as the question of inclusion of γ in C. Therefore the problem is reduced to characterizing the set C.

In general, a set $A \subseteq \mathbb{R}^k$ is called a convex cone iff for every $\mathbf{v}_1, \mathbf{v}_2 \in A$ and $\lambda_1, \lambda_2 \geq 0$, it follows that $\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \in A$. The smallest convex cone containing a given set $S \subset \mathbb{R}^k$ is known as the conical hull of S and is given by

$$\text{conic.hull}(S) = \{\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_l \mathbf{u}_l : \ \lambda_i \geq 0, \ \mathbf{u}_i \in S\}$$
(1.5)

It is straightforward to see that the set of moments C is a convex cone and is known as the moment cone. The study of the so-called dual cones provides a way to check inclusion of a vector in a convex cone.

1.3   Dual cones and the bipolar theorem

In this section $A \subset \mathbb{R}^n$ represents a general convex cone.

Definition 1 (Dual of a cone): If $A \subset \mathbb{R}^n$ is a convex cone, its dual, $A^* \subset (\mathbb{R}^n)^*$ is defined as

$$A^* = \{f : \quad f \in (\mathbb{R}^n)^*, f(\mathbf{v}) \geq 0 \ \forall \ \mathbf{v} \in A\}$$
(1.6)

Here, $(\mathbb{R}^n)^*$ is the set of all linear functionals on $\mathbb{R}^n$. We we illustrate this definition with two examples.

Example-1: Let $A = \{(x, 0) : \ x \geq 0\} \subset \mathbb{R}^2$. This is the positive half of the x-axis. By definition, $A^* = \{f : \ f \in (\mathbb{R}^2)^*, f(x, 0) \geq 0 \ \forall \ x \geq 0\}$. A linear functional f on $\mathbb{R}^2$ is also represented by a vector (a, b) and its action on $\mathbb{R}^2$ is by the inner product. Therefore,

3

$A^* = \{(a, b) : \ (a, b) \in (\mathbb{R}^2)^*, ax \geq 0 \ \forall \ x \geq 0\} = \{(a, b) : \ a \geq 0\}$. Therefore, A^* is the right half of $(\mathbb{R}^2)^*$.

Example-2: Let $A = \text{conic.hull}\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l\} \subset \mathbb{R}^n$. That is, A is the cone generated by k linearly independent vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l$. It consists of vectors of the form $\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_l \mathbf{u}_l$, where $\lambda_i \geq 0$. We say that A is finitely generated. Denoting a functional f on $\mathbb{R}^n$ by a vector $\mathbf{v}$, the dual cone is defined as:

$$A^* = \{\mathbf{v} : \ \mathbf{v} \in \mathbb{R}^n, \ \mathbf{v} \cdot \mathbf{u}_j \geq 0 \text{ for } j = 1, 2, \cdots, l\}. \tag{1.7}$$

Note that if a vector $\mathbf{v}$ satisfies all of the inequalities $\mathbf{v} \cdot \mathbf{u}_i \geq 0$, then it also satisfies $\mathbf{v} \cdot \mathbf{u} \geq 0$ for every $\mathbf{u} \in A$. It is straightforward to see that A^* is also finitely generated, and its generators are the dual vectors of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l$, defined by the equations

$$\mathbf{u}_i \cdot \mathbf{v}_j = \delta_{ij} \tag{1.8}$$

Such vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_l$ exist because $\mathbf{u}_i$ are linearly independent. To show that $A^* = \text{conic.hull}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_l\}$, note that any vector of the form $\mathbf{v} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_l \mathbf{v}_l$ with $\lambda_i \geq 0$ satisfies $\mathbf{v} \cdot \mathbf{u}_i = \lambda_i \geq 0$. Further, any vector $\mathbf{v}$ can be written as a linear combination of $\mathbf{v}_i$ and the coefficients are given by $\mathbf{v} \cdot \mathbf{u}_i$. That is, $\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{v}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{v}_2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_l)\mathbf{v}_l \geq 0$. Therefore, if $\mathbf{v} \in A^*$, then by definition, $\mathbf{v} \cdot \mathbf{u}_i \geq 0$ and that implies $\mathbf{v}$ is a superposition of the $\mathbf{v}_i$'s with non-negative coefficients and therefore, it is in the conical hull of the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_l\}$. Thus, $A^* = \text{conic.hull}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_l\}$.

This example illustrates the utility of a dual cone. In this particular example. a vector $\mathbf{u}$ can be checked for inclusion in A by verifying a finite number of inequalities. That is, $\mathbf{u} \in C$ iff $\mathbf{u} \cdot \mathbf{v}_i \geq 0$ for $i = 1, 2, \cdots, k$. However, in general, the dual cone may not be finitely generated. Nevertheless, dual cone of the moment cone is useful in solving TKMPs. In theorem 1.4.1, we show that the dual of the moment cone is the cone of non-negative polynomials after defining the latter and describing some of its properties.

1.4   The cone of non-negative polynomials

Let $I \subset \mathbb{N}^k$ be a finite index set with n elements. Each element of I represents a monomial. The space of all polynomials obtained by a superposition of these n monomials is an n dimensional vector space. It includes only those polynomials that consist of monomials that appear in I. Every such polynomial can be represented by an n dimensional vector. That is, $P(\mathbf{x}) = P(\mathbf{x}) = a_1\mathbf{x}^{\mathcal{N}_1} + a_2\mathbf{x}^{\mathcal{N}_2} + \cdots + a_n\mathbf{x}^{\mathcal{N}_n}$ can be represented as the vector $\mathbf{a} = (a_1, a_2, \cdots, a_n)$. The subset of this vector space consisting of polynomials that are non negative on $K \subseteq \mathbb{R}^k$ is a convex cone and is known as the cone of non-negative polynomials, denoted by C'.

$$C' = \left\{ P(\mathbf{x}) : \ P(\mathbf{x}) = a_1\mathbf{x}^{\mathcal{N}_1} + a_2\mathbf{x}^{\mathcal{N}_2} + \cdots + a_n\mathbf{x}^{\mathcal{N}_n}, \ P(\mathbf{x}) \geq 0 \ \forall \ \mathbf{x} \in K \right\}. \quad (1.9)$$

To show that C' is a convex cone, consider two polynomials $P_1(\mathbf{x}), P_2(\mathbf{x}) \in C'$ and two non-negative real numbers $\lambda_1, \lambda_2 \geq 0$. The polynomials $P_1(\mathbf{x})$ and $P_2(\mathbf{x})$ contain only those monomials that appear in I and so does the polynomial $\lambda_1 P_1(\mathbf{x}) + \lambda_2 P_2(\mathbf{x})$. Further more, for any $\mathbf{x} \in K$, $P_1(\mathbf{x}) \geq 0$ and $P_2(\mathbf{x}) \geq 0$ and therefore, $\lambda_1 P_1(\mathbf{x}) + \lambda_2 P_2(\mathbf{x}) \geq 0$. Thus, the polynomial $\lambda_1 P_1(\mathbf{x}) + \lambda_2 P_2(\mathbf{x})$ is also in C'. Theorem 1.4.1 asserts that the closure of C' is the dual cone of the cone of moments C. Before proving this theorem, we illustrate the cone of non-negative polynomials with two examples.

Example-3: Let us consider the simple case where $k = 1$, $K = \mathbb{R}$ and $I = \{0, 1, 2\}$. That is, the relevant monomials are $1, x$ and x^2. The resulting cone is the set of all globally non-negative quadratics:

$$C' = \left\{ P(x) = a_0 + a_1 x + a_2 x^2 : \ P(x) \geq 0 \ \forall \ x \in \mathbb{R} \right\} \quad (1.10)$$

A quadratic $a_0 + a_1 x + a_2 x^2$ is globally non-negative iff $a_2 \geq 0$ and $a_1^2 \leq 4a_0 a_2$. Therefore, The cone C' is described by these two inequalities. Fig. 1 shows a cross section of this

cone at $a_0 = 1$.

Example-4: Let us now consider the cone of non-negative multivariate quadratic polynomials. For $m > 1$ let us assume I includes indices corresponding to monomials $\{1, x_1, \cdots, x_k, x_1^2, x_1 x_2, x_1, x_3, \cdots, x_i x_j, \cdots x_k^2\}$. That is, C' includes all polynomials of the form $P(\mathbf{x}) = a_0 + 2a_1 x_1 + \cdots + 2a_k x_k + a_{11} x_1^2 + 2a_{12} x_1 x_2 \cdots + a_k x_k^2$ that are non-negative on $\mathbb{R}^k$. It convenient to arrange the coefficients in the form of a matrix A such that $P(\mathbf{x}) = \mathbf{x} A \mathbf{x}^T$, where $\mathbf{x}^T$ is the transpose of the row vector $\mathbf{x}$ and A is given by

$$
A = \begin{pmatrix} a_0 & a_1 & \cdots & a_k \\ a_1 & a_{11} & \cdots & a_{1k} \\ \vdots & & \ddots & \vdots \\ a_k & a_{1k} & \cdots & a_{kk} \end{pmatrix}.
\tag{1.11}
$$

The cone $C' = \{A : \mathbf{x} A \mathbf{x}^T \geq 0 \ \forall \ \mathbf{x} \in \mathbb{R}^k\}$ is the set of all positive semidefinite matrices A, also known as the *PSD cone*. This follows from the fact that $\mathbf{x} A \mathbf{x}^T \geq 0 \ \forall \ \mathbf{x} \in \mathbb{R}^k$ iff all of the eigenvalues of A are non-negative.

We are now ready to state and prove theorem 1.4.1.

Theorem 1.4.1 *For a given $K \subseteq \mathbb{R}^k$ and $I \subset \mathbb{N}^k$, let C be the moment cone and C' be the cone of non-negative polynomials. It follows that $C^* = cl.(C')$, where $cl.(C')$ is the closure of C'.*

Proof 1.4.1 *We begin by showing that $C' \subseteq C$. Let $P(\mathbf{x}) = a_1 \mathbf{x}^{\mathcal{N}_1} + \cdots + a_n \mathbf{x}^{\mathcal{N}_n} \in C'$. It follows that $P(\mathbf{x}) \geq 0$ for each $\mathbf{x} \in K$. Therefore, for any measure μ on K, it follows that $\int_K P(\mathbf{x}) d\mu = a_1 m_1 + \cdots + a_n m_n \geq 0$ implying that the vector $(a_1, \cdots, a_n)$ is in the dual of C. Thus, $C' \subseteq C^*$.*

Next, we show that $C^* \subseteq C'$. Let $(a_1, \cdots, a_n) \in C^*$. This means, for any measure μ on K, $a_1 \int_K \mathbf{x}^{\mathcal{N}_1} d\mu + \cdots + a_n \int_K \mathbf{x}^{\mathcal{N}_n} d\mu \geq 0$. We are to show that the corresponding polynomial $P(\mathbf{x}) = a_1 \mathbf{x}^{\mathcal{N}_1} + \cdots + a_n \mathbf{x}^{\mathcal{N}_n}$ is non-negative on K. For a point $\mathbf{x}_0 \in K$, we

define the moment $\mu_{\mathbf{x}_0}$ *as*

$$\mu_{\mathbf{x}_0}(B) = \begin{cases} 0 \text{ if } \mathbf{x}_0 \notin B \\ \\ 1 \text{ if } \mathbf{x}_0 \in B \end{cases} \tag{1.12}$$

for $B \subseteq K$. *Note that* $a_j \int_K \mathbf{x}^{\mathcal{N}_j} d\mu_{\mathbf{x}_0} = \mathbf{x}_0^{\mathcal{N}_j}$. *Therefore,* $P(\mathbf{x}_0) = a_1 \int_K \mathbf{x}^{\mathcal{N}_1} d\mu_{\mathbf{x}_0} + \cdots + a_n \int_K \mathbf{x}^{\mathcal{N}_n} d\mu_{\mathbf{x}_0} \geq 0$. *Thus,* $P(\mathbf{x})$ *is non-negative on* K *and therefore,* $C^* \subseteq C'$.

This result can be used to solve the TMP. In particular, the cone of non-negative polynomials can be used to develop criteria to test whether a given vector is in the moment cone. We illustrate this idea with two examples.

Example-5: Let us again consider the simple case where $k = 1$, $K = \mathbb{R}$ and $I = \{0, 1, 2\}$. The resulting moment cone consists of all three dimensional vectors $\mathbf{m} = (m_0, m_1, m_2)$ such that for some measure μ, $m_0 = \int_K d\mu$, $m_1 = \int_K x d\mu$ and $m_2 = \int_K x^2 d\mu$.

$$C = \{\mathbf{m} = (m_0, m_1, m_2) : \ m_i = \int_K x^i d\mu\}. \tag{1.13}$$

The corresponding truncated moment problem is to determine whether a given vector $\gamma = (\gamma_0, \gamma_1, \gamma_2)$ is in C. An obvious necessary condition comes from the Cauchy-Schwartz inequality — γ is in C only if $\gamma_0 \gamma_2 \geq \gamma_1^2$. Below, we show, using theorem 1.4.1, that this condition is also sufficient. The dual of C is the cone of non-negative quadratics discussed in example-3:

$$C' = \{\mathbf{a} = (a_0, a_1, a_2) : \ a_2 \geq 0 \text{ and} a_1^2 \leq 4a_0 a_2 \} \tag{1.14}$$

The extreme point of C' are precisely those vectors that satisfy $a_1^2 = 4a_0 a_2$ with $a_2 \geq 0$. Therefore, C' can be written as the conical hull of its extreme points:

$$C' = \text{conic.hull}\{\mathbf{a} = (a_0, a_1, a_2) : \ a_2 \geq 0 \text{ and } a_1^2 = 4a_0 a_2 \} \tag{1.15}$$

Following theorem 1.4.1, a vector γ is in C iff $\gamma_0 a_0 + \gamma_1 a_1 + \gamma_2 a_2 \geq 0$ for each $\mathbf{a}$ in the generating set of C', i.e., the criterion is $\gamma_0 a_0 + \gamma_1 a_1 + \gamma_2 a_1^2/(4a_0) \geq 0$ for every a_0, a_1 with

$a_0 \geq 0$. In other words, γ is in C iff $\gamma_0 \geq 0$ and $\gamma_0\gamma_2 \geq \gamma_1^2$.

Example-6: We next consider the multivariate generalization of the previous example. similar to what was considered in example-4. For $m > 1$ and $K = \mathbb{R}^k$, let us assume I includes indices corresponding to monomials $\{1, x_1, \cdots, x_k, x_1^2, x_1x_2, x_1, x_3, \cdots, x_ix_j, \cdots x_k^2\}$. The correspondimg moments can be arranged into a matrix, similar to the matrix V in example-4. Defining $m_0 = \int_K d\mu$, $m_i = \int_K x_i d\mu$ and $m_{ij} = \int_K x_ix_j d\mu$, we may arrange these moments in a matrix M

$$
M = \begin{pmatrix} m_0 & m_1 & \cdots & m_k \\ m_1 & m_{11} & \cdots & m_{1k} \\ \vdots & & \ddots & \vdots \\ m_k & m_{1k} & \cdots & m_{kk} \end{pmatrix}. \tag{1.16}
$$

The moment cone C is the cone of all such matrices. The corresponding moment problem is to determine if a given matrix Γ is in C. The dual of C is the cone of non-negative polynomials described in example-4, i.e., the PSD cone. By theorem 1.4.1, $\Gamma \in C$ iff $\text{Tr}(\Gamma A) \geq 0$ for each $A \in C'$, i.e., for each positive semidefinite A. In particular, for an arbitrary $\mathbf{x} \in \mathbb{R}^k$, $\mathbf{x}^T\mathbf{x} \in C'$, i.e., $\mathbf{x}^T\mathbf{x}$ is PSD and therefore, $textTr(\Gamma\mathbf{x}^T\mathbf{x}) = \mathbf{x}\Lambda\mathbf{x}^T \geq 0$ is a necessary condition for Γ to be in C. In other words, Λ has to be PSD. It follows that this criterion is also sufficient, because $\text{Tr}(V\Gamma) \geq 0$ when both A and Γ are PSD. Therefore, the PSD cone is self dual.

1.5 Semialgebraic sets

All examples of moment cones considered so far are characterized by polynomial inequalities. That is, the criteria to determine whether a given set of numbers are moments have always been a set of polynomial inequalities. In example-4, the criteria was $\gamma_0 \geq 0$ and $\gamma_0\gamma_2 - \gamma_1^2 \geq 0$. In example-6, the criterion was $\Gamma \succeq 0$, which can be rephrased as a set of polynomial inequalities, each of which can be written as non-negativity of a polynomial

in the elements of Γ. The extreme points of such cones are defined by the zeros of these polynomials.

Sets defined by non-negativity of a finite collection of polynomials are known as *basic semi-algebraic* sets. Sets that can be described by unions, intersections and projections of basic semi-algebraic sets for a larger class known as *semi-algebraic sets*.

In general, any cone of polynomials non-negative on a semi-algebraic set is semi-algebraic. Further, duals of semi-algebraic sets are also semi-algebraic, which implies that moments cones are semi-algebraic as well.

CHAPTER 2

ENTANGLEMENT AND RELATION TO TKMP

Complimentary to the previous chapter, in this chapter, we provide an introduction to entanglement, also aimed at the non-expert audience. This chapter also serves as a motivation to the main problem considered in the thesis, which we define in the next chapter. In section 2.1 we develop the basic ideas of "states" and "observable parameters". In section 2.2, we describe the pivotal features of quantum mechanics that lead to the phenomena of entanglement. In section 2.3 we define entanglement and discuss how it is related to TKMPs. Finally, in section 2.4, we discuss a standard example, illustrating the connection between entanglement and TKMP.

2.1 Physical Systems and Observable Parameters

A physical system is characterized by experimentally measurable parameters. For example, a pendulum has three experimentally measurable parameters – the x, y and z coordinates of its bob. If the bob is hanging down via a rigid support of length R, the parameters are constrained to a sphere of radius R, i.e., $x^2 + y^2 + z^2 = R^2$. If the bob hanging down via a thread of length R, the parameters are confined to a ball of radius R, i.e., $x^2 + y^2 + z^2 \leq R^2$. In general, an experimentally measurable parameter is represented by a real number and a set of n parameters of a physical system is represented by a vector in $\mathbb{R}^n$. A physically reasonable assumption is that the domain $\mathcal{D}$ to which these vectors are confined is compact. We use the symbols Γ_i to represent the name of a parameter and x_i to represent its measured value. For instance, in the above example of the pendulum, Γ_1 would represent "x-coordinate" and x_1 would represent a measured value of the x-coordinate. This sounds redundant, but in physics not every parameter is a physical coordinate; for instance there are parameters such as "mass", "charge", "momentum" etc and therefore we need to

distinguish between the name and the value.

One can assume that the space $\mathcal{D}$ indeed represents the set of configurations in which the physical system can be prepared. This however, is not accurate within the framework of quantum mechanics, because of the unique features of a measurement in quantum mechanics, which we describe in the following section.

2.2 Measurement in Quantum Mechanics

What sets aside quantum mechanics is the way in which the values x_i of the parameters Γ_i are obtained. Measurement in quantum mechanics involves a random event that also affects the system's configuration. If a system is initiated in some configuration and a parameter Γ_i is measured, this measurement returns a random number γ_i and changes the system's configuration to a random configuration. The random event in the measurement results in the random number γ_i and a random final configuration of the system. If the system is reset into its original configuration, and the same parameter Γ_i is measured, we obtain a second random number γ_i' and the system gets put into another random configuration. In this sense, measurement in quantum mechanics is "destructive" i.e., it demolishes the system's configuration. If this process is repeated ν times, while resetting the system to its original configuration each time, we obtain a set of random numbers $\gamma_i(1), \gamma_i(2), \cdots, \gamma_i(\nu)$. The measured value, x_i of the parameter Γ_i is defined as

$$x_i = \lim_{\nu \to \infty} \frac{1}{\nu} \sum_{k=1}^{\nu} \gamma_i(k) \tag{2.1}$$

The measured value refers to this limit and $\mathcal{D}$ is the space of $\mathbf{x} = (x_1, \cdots, x_n)$, where each x_i is defined as above.

We reiterate the two crucial features of this measurement process.

(i) Each coordinate of a point in $\mathcal{D}$ is the mean value of a large number of outcomes obtained in independent measurements.

(ii) Obtaining each outcome $\gamma_i(k)$ demolishes the system's configuration and therefore allows for no subsequent measurement without resetting the system's configuration.

The above features have strong consequences on the nature of the space $\mathcal{D}$. For instance, (i) implies that $\mathcal{D}$ is convex. If $\mathbf{x}, \mathbf{x}' \in \mathcal{D}$, we may include a random event in the process of initiating the system, that prepares the system in the configuration corresponding to $\mathbf{x}$ with probability λ and in the configuration corresponding to $\mathbf{x}'$ with probability $1 - \lambda$. The measured parameter values would then be $\lambda \mathbf{x} + (1 - \lambda)\mathbf{x}'$. Therefore, $\mathcal{D}$ should be convex. It can be viewed as the section of a cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ defined as

$$\mathcal{C} = \{\mathbf{x} = (x_0, x_1, \cdots, x_n) : \ x_0 > 0 \ \& \ (x_1/x_0, \cdots, x_n/x_0) \in \mathcal{D}\} \tag{2.2}$$

It will be clear later that it is convenient to use the cone instead of its section. We formulate the rest of the ideas in terms of the cone $\mathcal{C}$.

The consequences of (ii) are deeper. In particular, the correlations coming from the random event in a measurement are inaccessible. For instance, one cannot measure the correlations between Γ_i and Γ_j, i.e., $\frac{1}{\nu} \sum_k \gamma_i(k)\gamma_j(k)$ is not a correlation, because there is no correspondence between $\gamma_i(k)$ and $\gamma_j(k)$. They are the results of two independent random events.

2.3 Non-locality and Entanglement

A fundamental question prompted by (ii) is, if a system is prepared in some configuration and one measurement of Γ_i is made, resulting in an outcome $\gamma_i(k)$ what can be said about the corresponding value of parameter Γ_j for some $j \neq i$?

There are two possible propositions that answer this question:

Proposition A: The value of Γ_j corresponding to $\gamma_i(k)$ is undefined.

Proposition B: The value of Γ_j corresponding to $\gamma_i(k)$ is well defined but *hidden*, i.e., not experimentally accessible.

The two propositions would lead to two different ways of modeling a measurement. Under proposition A, we can model a measurement of Γ_i as a random process described by a measure μ_i defined on $\pi_i(\mathcal{C})$, where $\pi_i : \mathcal{C} \to \mathbb{R}$ is the projection to the $i-$th coordinate. The outcome $\gamma_i(k) \in \pi_i(\mathcal{C})$ would be a random number whose distribution is given by μ_i. Under this proposition, the measures μ_i, μ_j are in general unrelated — they do not necessarily come from a common measure over $\mathcal{C}$. Under proposition B, we can model a measurement in a similar way using measures μ_i, with the additional condition that they all come from a common measure μ defined on $\mathcal{C}$. That is, there exists a measure μ defined on $\mathcal{D}$ such that for every $X \subset \pi_i(\mathcal{C})$, μ_i satisfies $\mu_i(X) = \int_{\pi_i^{-1}(X)} d\mu$.

A more fundamental difference between the two propositions is revealed when we consider a composite system, consisting of two physically separate subsystems. Let us consider two physical systems, whose parameters are $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ and $\Sigma_1, \Sigma_2, \cdots, \Sigma_m$ and domains $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. We denote the coordinates of points in $\mathcal{C}_1$ as $\mathbf{x} = (x_0, x_1, \cdots, x_n)$ and in $\mathcal{C}_2$ as $\mathbf{y} = (y_0, y_1, \cdots, y_m)$. We may consider each of the systems as subsystems of a composite system, consisting of both of them. The parameters of this composite system include $\Gamma_1, \Gamma_2, \cdots, \Gamma_n, \Sigma_1, \Sigma_2, \cdots, \Sigma_m$ and the products $\Gamma_i \Sigma_j$, whose values are defined as $\lim_{\nu \to 0} \frac{1}{\nu} \sum_k \gamma_i(k) \sigma_j(k)$. The two systems may be physically separated and therefore we can presume that Γ_i and Σ_j can be measured simultaneously. It is convenient to introduce symbols Γ_0 and Σ_0 as placeholders for "no measurement" done on the two systems respectively. This way, $\Gamma_0 \Sigma_1$ would represent a measurement of Σ_1 on the second system. Thus, the set of parameters of the composite system may be listed as $\{\Gamma_i \Sigma_j : i = 0, 1, \cdots, n \ \& \ j = 0, 1, \cdots, m\}$. Therefore the composite system has $(n+1)(m+1) - 1$ parameters and the corresponding domain is a cone $\mathcal{C}_{12} \subset \mathbb{R}^{(n+1)(m+1)}$. We denote the points in $\mathcal{C}_{12}$ as $\mathbf{z} = (z_{00}, z_{10}, z_{20}, \cdots, z_{n0}, z_{01}, \cdots, z_{0m}, z_{11}, \cdots, z_{nm})$. Here, z_{ij} is a measured value of $\Gamma_i \Sigma_j$ and z_{00} is the scale factor that appears in the cone $\mathcal{C}_{12}$. A single measurement of the parameter $\Gamma_i \Sigma_j$ is described by a measure μ_{ij} defined on $\pi_i^{-1}(\mathcal{C}_1) \times \pi_j^{-1}(\mathcal{C}_2)$.

Under proposition B, this domain has a simple characterization in terms of moments of measures defined over $\mathcal{C}_1 \times \mathcal{C}_2$. If μ and μ' are measures defined over $\mathcal{C}_1 \times \mathcal{C}_2$ we say that $\mu \sim \mu'$ iff $\int x_i y_j d\mu = \int x_i y_j d\mu'$ for $i = 0, 1, \cdots, n$ and $j = 0, 1, \cdots, m$. Under proposition B, $\mathcal{C}_{12}$ can be constructed as the quotient of the space $M(\mathcal{C}_1 \times \mathcal{C}_2)$ of all measures μ over $\mathcal{C}_1 \times \mathcal{C}_2$:

$$\mathcal{C}_{12} = M(\mathcal{C}_1 \times \mathcal{C}_2)/ \sim \tag{2.3}$$

This follows from the fact that under proposition B, z_{ij} is the moment of μ_{ij} and the latter comes from a global measure μ on $\mathcal{C}_1 \times \mathcal{C}_2$. Therefore, every set of measured values $\{z_{ij}\}$ must satisfy $z_{ij} = \int_{\mathcal{C}_1 \times \mathcal{C}_2} x_i x_j d\mu$ for some measure μ.

Under proposition A, $\mathcal{C}_{12} \supseteq M(\mathcal{C}_1 \times \mathcal{C}_2)/ \sim$, simply because every measure μ defined on $\mathcal{C}_1 \times \mathcal{C}_2$ has corresponding measured μ_{ij} defined on $\pi_i(\mathcal{C}_1) \times \pi_j(\mathcal{C}_2)$, but not every set of measures μ_{ij} come from a common measure μ. Thus, proposition A and B imply different domains of parameters of a composite system. A point $\mathbf{z}$ outside the cone $M(\mathcal{C}_1 \times \mathcal{C}_2)/ \sim$ are significant in two ways. First, if observed experimentally, such points can invalidate proposition B. Second, such points represent a new kind of correlation between the two systems.

As we mentioned before, the whole process of initializing the system in a configuration and measuring its parameters involves two random events — one during the initializing and the other during measurement. Points inside the cone $M(\mathcal{C}_1 \times \mathcal{C}_2)/ \sim$ represent correlations during the first random event and points outside this cone represent correlations during the second random event.

More precisely, points inside $M(\mathcal{C}_1 \times \mathcal{C}_2)/ \sim$ can be modeled by a global measure μ defined over $\mathcal{C}_1 \times \mathcal{C}_2$ that describes the first random event, i.e., the initialization and a pair of local measures μ_1 and μ_2 defined over $\mathcal{C}_1$ and $\mathcal{C}_2$ that describe the second random event, i.e., the measurement, thereby avoiding any correlation during the measurement. This way, such points can be explained without postulating any correlation at the measurement stage. However, points outside $M(\mathcal{C}_1 \times \mathcal{C}_2)/ \sim$ can only be described by a measure μ defined

over $\mathcal{C}_1 \times \mathcal{C}_2$, describing the first random event and a set of measures μ_{ij} defined over $\pi_i(\mathcal{C}_1) \times \pi_j(\mathcal{C}_2)$ describing the second random event. It is in general, impossible to avoid postulating a correlation during the measurement since this correlation is described by the measures μ_{ij}.

A correlation in the second random event, i.e., during the measurement, is counter intuitive — it means that the act of measuring Γ_i on the first system can have a physical effect on the second system, as described by μ_{ij}. Note that no assumption has been made so far regarding the physical separation between the two systems at the time of the measurement — they may be separated by a very large distance, potentially in different parts of the world. Therefore such correlations are not bound by physical proximity. They are known as *non-local correlations*. Configurations of the two systems that allow for non-local correlations are called *entangled* states.

To summarize, proposition A naturally leads to counter-intuitive non-local correlations, while under proposition B, we can maintain locality in all of the correlations. The inevitable appearance of non-local correlation under proposition A is also known as *EPR paradox*, named after Einstein, Podolsky and Rosen, who pointed it out in a well known paper [4]. Quantum mechanics favors proposition A, while Albert Einstein favored proposition B, quoting the EPR paradox as a reason to discard proposition A. In the same paper, he also proposed that quantum mechanics can be expanded by appending variables which tell us the values Γ_j after a measurement of Γ_i and the resulting theory would favor proposition B and will therefore restore locality of correlations. The appended variables are known as hidden variables a theory that expands out quantum mechanics this way is called a *local hidden variable theory*.

The straightforward verification of Einstein's proposal is to check if $\mathcal{C}_{12} = M(\mathcal{C}_1 \times \mathcal{C}_2)/ \sim$ under quantum mechanics. In [5] in 1961, J. S. Bell derived the first necessary criterion for membership in the cone $M(\mathcal{C}_1 \times \mathcal{C}_2)/ \sim$ for a simple system consisting of a pair of two-level atoms, also known as *Bell inequality*. He showed that according to

quantum mechanics, there is at least one point in $\mathcal{C}_{12}$ that violates Bell inequality, i.e., is not included in $M(\mathcal{C}_1 \times \mathcal{C}_2)/\sim$, thus invalidating proposition B and Einstein's proposal. The first experimental test was conducted by Alian Aspect in 1981 [6], where, an experimentally obtained set of measured values was shown to represent a point outside the cone $M(\mathcal{C}_1 \times \mathcal{C}_2)/\sim$. Such experiments are called Bell test experiment.

A typical Bell test experiment would involve (i) preparing a set of two (or more) systems in a configuration that is expected to be entangled. (ii) Experimentally measuring a set of parameters **z** of the composite system. And (iii) Proving that the measured set of parameters **z**, is point outside the cone $M(\mathcal{C}_1 \times \mathcal{C}_2)/\sim$. The last step involves a characterization of the moment cone $M(\mathcal{C}_1 \times \mathcal{C}_2)/\sim$. While such tests have been performed on multiple systems, the practical utility of such non-local correlations has not been fully explored. Theoretically, such correlations have found applications in quantum computation[], quantum communication and quantum metrology.

In this work, we consider a system consisting of N subsystems and characterize the corresponding cone $M(\mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_N)/\sim$, thereby deriving membership criteria for this cone. The results in this work will enable demonstrating non-local correlations among N subsystems.

2.4   Example

In this section we illustrate the ideas presented in the previous three sections using the simple case of a pair of two level atoms. The purpose of this section is to show, to the physics audience how the model described above applies to physical systems and formulate the well known problem of bipartite mixed state entanglement as a truncated K-moment problem. This section assumes prior knowledge of density matrices.

We consider a composite system of a pair of two-level atoms. A general mixed state of such a system is represented by a 4×4 density matrix. We use superscripts A and B to indicate the two subsystems and we use ρ^{AB} to represent the density matrix of the

composite system.

A mixed state represented is said to be separable, iff the corresponding density matrix ρ_S^{AB} can be written as an incoherent superposition of product states:

$$\rho_S^{AB} = \sum_{i=1}^{n} w_i \rho_i^A \otimes \rho_i^B \tag{2.4}$$

The subscript S is used to indicate that the state is separable. ρ_i^A and ρ_i^B are 2×2 density matrices representing mixed states of the subsystems A and B respectively. $w_i > 0$ are weights satisfying $\sum_{i=1}^{n} w_i = 1$. A density matrix that does not admit such a resolution is called an entangled state.

A 2×2 density matrix is represented by a point inside the Bloch ball, $\mathbb{B}$. The later is the unit ball in $\mathbb{R}^3$. Indeed, a 2×2 density matrix can be written as a superposition of the Pauli matrices

$$\rho = \frac{1}{2}(\mathbf{1} + \mathbf{u} \cdot \sigma) = \frac{1}{2}(\mathbf{1} + u_x \sigma_x + u_u \sigma_y + u_z \sigma_z) \tag{2.5}$$

Here, $\mathbf{u} = (u_x, u_y, u_z)$ is a point in $\mathbb{B}$. The 2×2 identity matrix is represented by $\mathbf{1}$. σ_x, σ_y and σ_z are the Pauli matrices defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6}$$

Therefore the definition of a separable state can be expressed as

$$\rho_S^{AB} = \frac{1}{4} \sum_{i=1}^{n} w_i (\mathbf{1} + \mathbf{u}_i \cdot \sigma^A) \otimes (\mathbf{1} + \mathbf{v}_i \cdot \sigma^B) \tag{2.7}$$

σ^A and σ^B are the Pauli pseudo vectors corresponding to subsystems A and B respectively. An equivalent definition of separable states is obtained by replacing the sum by an integral and the weights w_i by a measure μ defined over $\mathbb{B} \times \mathbb{B}$. That is, a state is separable iff there exists a measure μ defined over $\mathbb{B} \times \mathbb{B}$ such that the corresponding density matrix ρ_S^{AB} can

be written as

$$\rho_S^{AB} = \frac{1}{4} \int_{\mathbb{B} \times \mathbb{B}} d\mu (\mathbf{1} + \mathbf{u} \cdot \sigma^A) \otimes (\mathbf{1} + \mathbf{v} \cdot \sigma^B) \tag{2.8}$$

The integral is carried out over $(\mathbf{u}, \mathbf{v}) \in \mathbb{B} \times \mathbb{B}$.

The Pauli basis for 2×2 density matrices can be extended to 4×4 density matrices. Indeed every 4×4 density matrix can be written as

$$\rho^{AB} = \frac{1}{4} \left(\mathbf{1} \otimes \mathbf{1} + \mathbf{p} \cdot \sigma^A \otimes \mathbf{1} + \mathbf{1} \otimes \sigma^B \cdot \mathbf{q} + \sum_{\alpha, \beta = x, y, z} t_{\alpha\beta} \sigma_\alpha^A \otimes \sigma_\beta^B \right) \tag{2.9}$$

Here, $\mathbf{p}, \mathbf{q} \in \mathbf{R}^3$ represent the vector polarization and t is a 3×3 matrix representing the correlations. Indeed, the components of $\mathbf{p} = (p_x, p_y, p_z)$, $\mathbf{q} = (q_x, q_y, q_z)$ and t are the expectation values of the respective observables.

$$
\begin{aligned}
p_\alpha &= \mathrm{Tr}(\rho^{AB} \sigma_\alpha^A \otimes \mathbf{1}) \\
q_\beta &= \mathrm{Tr}(\rho^{AB} \mathbf{1} \otimes \sigma_\beta^B) \\
t_{\alpha\beta} &= \mathrm{Tr}(\rho^{AB} \sigma_\alpha^A \otimes \sigma_\beta^B)
\end{aligned}
\tag{2.10}
$$

For $\alpha, \beta = x, y, z$. Every 4×4 density matrix is uniquely characterized by this set of observable expectation values. The definition of a separable state can be formulated in terms of these expectation values. A state ρ^{AB} with parameters $\mathbf{p}$, $\mathbf{q}$ and t is separable iff there exists a measure μ defined over $\mathbb{B} \times \mathbb{B}$ such that

$$
\begin{aligned}
p_\alpha &= \int_{\mathbb{B} \times \mathbb{B}} u_\alpha d\mu \\
q_\beta &= \int_{\mathbb{B} \times \mathbb{B}} v_\beta d\mu \\
t_{\alpha\beta} &= \int_{\mathbb{B} \times \mathbb{B}} u_\alpha v_\beta d\mu
\end{aligned}
\tag{2.11}
$$

where u_α are the components of the vector $\mathbf{u} \in \mathbb{B}$ and v_β are the components of the vector $\mathbf{v} \in \mathbb{B}$, that are integrated over. In other words, the state ρ^{AB} is separable iff $\mathbf{p}$, $\mathbf{q}$ and t are

18

the moments of some measure μ defined on $\mathbb{B} \times \mathbb{B}$. This follows from the following simple observation:

$$(\mathbf{1} + \mathbf{u} \cdot \sigma^A) \otimes (\mathbf{1} + \mathbf{v} \cdot \sigma^B) = \mathbf{1} \otimes \mathbf{1} + \mathbf{u} \cdot \sigma^A \otimes \mathbf{1} + \mathbf{1} \otimes \sigma^B \cdot \mathbf{v} + \sum_{\alpha,\beta=x,y,z} u_\alpha v_\beta \sigma^A_\alpha \otimes \sigma^B_\beta.$$

(2.12)

Therefore, the problem of deciding whether a given density matrix ρ^{AB} is separable or entangled can be reformulated as a truncated k-moment problem. In the language of the previous section, u_α's are the x_i's , v_β's are the y_i's — they are the parameters of the two systems. The observables σ^A_α's correspond to Γ_i's and σ^B_β's correspond to Σ_i's. The parameters p_α, q_β and $t_{\alpha\beta}$ correspond to the z_{ij}'s. This particular example has a complete solution and is known as the *positive partial transpose* criterion [7]. In fact this criterion can be recovered by characterizing the relevant cone of non-negative polynomials.

CHAPTER 3

THE PROBLEM STATEMENT

In this chapter, we state the main problem considered in this work together with a brief discussion of the specific physical system by which this problem is inspired.

The physical system of relevance for this problem consists of n atoms, which in a typical experiment takes values between 500 and 10^5. However, the specific value of n is not important in this problem. The domain $\mathcal{D}$ of parameters for each atom is a unit ball $\mathbb{B} = \{\mathbf{x} : \ \mathbf{x} \in \mathbb{R}^3 \ \& \ ||\mathbf{x}|| \leq 1\}$ in $\mathbb{R}^3$. We consider the TKMP that is equivalent to the problem of detecting entangled states of this system. As discussed in the previous chapter, the relevant "K" in this TKMP is the set $\mathbb{B} \times \mathbb{B} \times \cdots \times \mathbb{B}$. We would be studying the moment cone generated by measures defined over this set.

An additional feature of the system under consideration is that their physical properties remain invariant under the permutation of the n atoms. Such atoms are called *Identical Bosons* and the specific system of n Bosons that we are considering is called a *Bose-Einstein Condensate* (BEC). Therefore, we restrict ourselves to the moment cone generated by measures over $\mathbb{B} \times \mathbb{B} \times \cdots \times \mathbb{B}$ that are invariant under the action of the symmetric group S_n. Also, an experimental limitation is set on the order of accessible moments. We restrict ourselves to quadratic moments.

We now formulate the problem inspired from the above described physical system. Let $D^d \subset \mathbb{R}^d$ be the closed unit ball in a d dimensional Real space. That is, $D^d = \{\mathbf{x} : \ \mathbf{x} \in \mathbb{R}^d, \ ||\mathbf{x}|| \leq 1\}$. We define K_n^d as the product of n such unit balls:

$$K_n^d = D^d \times \cdots \times D^d \subset \mathbb{R}^{nd}$$

It is a compact, full dimensional subset of $\mathbb{R}^{nd}$. For the specific physical system described

above, $d = 3$. However, we let d be a variable in the rest of the thesis. Points in this space can be represented by a n-tuple of $d-$dimensional vectors $(\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_n)$, where, $\mathbf{x}_i = (x_{i,1}, x_{i,2} \cdots x_{i,d}) \in D^d$. The symmetric group S_n(i.e., the group of all permutations of the set $\{1, 2, \cdots n\}$) acts on K_n^d by permuting the vectors $\mathbf{x}_i$. For $\sigma \in S_n$, its action on K_n^d is given by $\sigma \circ (\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_n) = (\mathbf{x}_{\sigma(1)}, \mathbf{x}_{\sigma(2)}, \cdots \mathbf{x}_{\sigma(n)})$. We refer to a measure μ defined on K_n^d as a *symmetric measure* if it is invariant under the action of S_n. That is, if $A \subset K_n^d$,

$$\mu(A) = \mu(\sigma \circ A), \ \forall \, \sigma \in S_n.$$

In this work, we consider the Truncated K-moment problem for symmetric measures over K_n^d. Moments of symmetric measures are also invariant under coordinate permutations.

Let $V_{n,d}$ the vector space of real square-free polynomials in $d \cdot n$ variables $\mathbf{x_1}, \ldots, \ldots, \mathbf{x_n}$, $\mathbf{x_i} = (x_{i,1}, \ldots, x_{i,d})$ which is spanned by the following symmetric polynomials:

$$1, \quad s_\alpha = \sum_{i=1}^{n} x_{i,\alpha}, \ \ 1 \leq \alpha \leq d$$

and

$$s_{\alpha\alpha} = \sum_{1 \leq i < j \leq n} x_{i,\alpha} x_{j,\alpha}, \ \ 1 \leq \alpha \leq d.$$

The dimension of $V_{n,d}$ is therefore $2d + 1$.

We use m_0, m_α and $m_{\alpha\alpha}$ to denote the corresponding moments:

$$m_0 = \int_{K_n^d} 1 \, d\mu, \quad m_\alpha = \int_{K_n^d} s_\alpha \, d\mu, \quad m_{\alpha\alpha} = \int_{K_n^d} s_{\alpha\alpha} \, d\mu.$$

The moment sequence $\mathbf{m} = (m_0, m_\alpha, m_{\alpha\alpha})$ lies in $\mathbb{R}^{2d+1}$. We define $C_{n,d} \subset \mathbb{R}^{2d+1}$ to be the set of all moment sequence coming from measures on K_n^d. Observe that $C_{n,d}$ is a closed convex cone because K_n^d is compact. The dual cone, $C_{n,d}^*$ consists of all symmetric

polynomials in $V_{n,d}$ non-negative on K_n^d:

$$C_{n,d}^* = \left\{ Q \in V_{n,d} : \ Q(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in K_n^d \right\}.$$

In this work, we are concerned with the characterization of the cone $C_{n,d}$ and/or its dual cone $C_{n,d}^*$.

CHAPTER 4

EXACT RESULTS

In this chapter, we provide some exact results to characterize the cone $C_{n,d}^*$ of non-negative polynomials. K_n^d is a compact, $n \cdot d$ dimensional manifold with boundary. Therefore, testing for membership of a quadratic form Q in $C_{n,d}^*$ would involve, by definition, checking for its non-negativity over an $n \cdot d$ dimensional manifold. In this chapter, we show that for $n \geq 2d$, there is a $2d(d-1)$ dimensional submanifold in K_n^d such that a quadratic form Q is non-negative on K_n^d if and only if it is non-negative on this submanifold. We refer to the sub-manifold as a *test manifold*, indicating that it can be used to check for membership of a quadratic in $C_{n,d}^*$. This is the main result of this chapter and is summarized in Theorem 5.2.1. Note that the dimension of the test manifold is independent of n. We also show that this theorem is *tight*, in the sense that there can not be a test manifold that is a smaller submanifold, for the special case with $d = 2$. The reduction in the dimension of the test manifold comes from the symmetry of the problem. We begin by illustrating this ides for the case of $d = 1$.

4.1   One Dimensional Case $d = 1$

As a simple first step we characterize the cones $C_{n,1}$ and $C_{n,1}^*$. The relevant unit ball is $D^1 = [-1, 1]$. A point in K_n^1 is represented by $(x_1, x_2, \cdots, x_n)$ where $x_i \in [-1, 1]$. Quadratic form $Q \in V_{n,1}$ is of the form:

$$Q = A_0 + A_1 s_1 + A_{11} s_{11}.$$

Note that Q has only linear terms in each variable x_i. Therefore, extreme values of Q occur when $x_i = \pm 1$. In other words, Q is non-negative on K_n^1 iff it is non-negative on the

hypercube $H_n = \{-1, +1\}^n$. For a point $\mathbf{x} = (x_1, \ldots, x_n) \in H_n$ with k entries -1's and $n - k$ entries $+1$'s we have $s_1(\mathbf{x}) = n - 2k$ and $s_{11}(\mathbf{x}) = (n - 2k)^2 - n$. We immediately obtain the following Proposition:

Proposition 4.1.1 *A polynomial* $Q = A_0 + A_1 s_1 + A_{11} s_{11} \in C^*_{n,1}$ *iff*

$$A_0 + A_1(n - 2k) + A_{11}((n - 2k)^2 - n) \geq 0,$$

holds for $k = 0, 1, \cdots, n$.

We see from the above that $C^*_{n,1}$ is a polyhedral cone. Figure 4.1 shows cross sections of $C^*_{n,1}$. It is straightforward to show that all of the $n + 1$ inequalities shown above are necessary to define $C^*_{n,1}$. Each inequality represents a side of the polygon shown in Figure 4.1. The dual cone $C_{n,1}$ of moment sequences coming from measures is also a polyhedral cone defined by $n+1$ inequalities. The defining inequalities of $C_{n,1}$ follow from Proposition 4.1.1:

Corollary 4.1.2 *A vector* $\mathbf{m} = (m_0, m_1, m_{11}) \in C_{n,1}$ *iff*

$$m_0(n - 1 + (n - 1 - 2k)^2) - m_1(n - 1 - 2k) + m_{11} \geq 0,$$

holds for $k = 0, 1, \cdots, n$.

Thus, when $d = 1$, $C_{n,1}$ and $C^*_{n,1}$ are both basic semi-algebraic, and are completely characterized by $n + 1$ linear inequalities.

4.2 General Dimension

When $d > 1$, $C_{n,d}$ is the conical hull of a semi-algebraic set. Indeed,

$$C_{n,d} = \text{Conic. Hull} \left\{ 1, s_1(\mathbf{x}), \cdots, s_d(\mathbf{x}), s_{11}(\mathbf{x}), \cdots, s_{dd}(\mathbf{x}) : \mathbf{x} \in K_n^d \right\}.$$
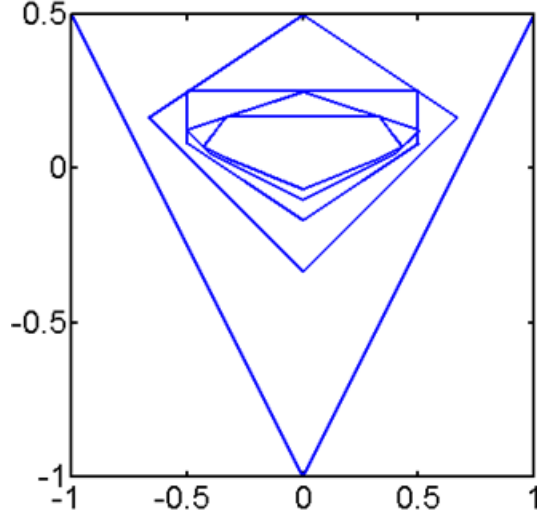
Figure 4.1: Cross sections of $C_{n,1}^{\prime *}$.

K_n^d is a basic semi-algebraic set and therefore, its image under a polynomial function is semi-algebraic. A polynomial $Q \in C_{n,d}^*$ is linear in each of its arguments and therefore, it is non-negative on K_n^d iff it is non-negative on its boundary, $\partial K_n^d = S^{d-1} \times \cdots \times S^{d-1}$. Therefore, membership of a polynomial Q in C_n^* is validated by verifying its non-negativity on an $n(d-1)$ dimensional manifold. However, in the following theorem, we show that it suffices to verify its non-negativity on finitely many copies $(O(n^{2d-1}))$ of a $2d(d-1)$ dimensional manifold. This theorem is an analogue of the degree principle [8, 9]. See also [10] and [11] for related results.

Theorem 4.2.1 $Q \in V_{n,d}$ *is non-negative on* $S^{d-1} \times \cdots \times S^{d-1}$ *if and only if* $Q(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ *is non-negative for all sets of* n *points* $\mathbf{x}_1, \cdots, \mathbf{x}_n$ *on* S^{d-1} *with only* $2d$ *of them distinct.*

Proof 4.2.1 *We will prove this Theorem using an elementary application of Lagrange multipliers. Recall that* $Q \in V_{n,d}$ *has the form*

$$Q = A_0 + \sum_{\alpha=1}^{n} A_\alpha s_\alpha + A_{\alpha\alpha} s_{\alpha\alpha}.$$

Let $\mathbf{x}^* = (\mathbf{x}_1^*, \cdots, \mathbf{x}_n^*)$ *be a global minimum of* Q *on* $\left(S^{d-1}\right)^n$ *and let* $\mathbf{x}_i^* = (\xi_{i,1}, \cdots, \xi_{i,d})$.

25

Since the global minimum is a critical point, it satisfies the following Lagrange multiplier equations:

$$A_\alpha + A_{\alpha\alpha}s_\alpha(\mathbf{x}^*) = \lambda_i + A_{\alpha\alpha}\xi_{i,\alpha}, \tag{4.1}$$

where λ_i are the Lagrange multipliers. Define

$$R_\alpha = A_\alpha + A_{\alpha\alpha}s_\alpha(\mathbf{x}^*).$$

There are nd Lagrange multiplier equations, and for a fixed i, there are d of them. We eliminate $\lambda_i, \xi_{i,2}, \cdots, \xi_{i,d}$ and obtain:

$$\xi_{i,\alpha} = \frac{R_\alpha \xi_{i,1}}{R_1 + (A_{\alpha\alpha} - A_{11})\xi_{i,1}}. \tag{4.2}$$

for $\alpha = 2, \cdots, d$. This expresses all the coordinates $\xi_{i,\alpha}$ in terms of one of them, $\xi_{i,1}$. Furthermore, the point $\mathbf{x}_i^*$ lies on S^{d-1}. Therefore,

$$\sum_{\alpha=1}^{d}\left(\frac{R_\alpha \xi_{i,1}}{R_1 + (A_{\alpha\alpha} - A_{11})\xi_{i,1}}\right)^2 = 1$$

This can be viewed as a polynomial of degree $2d$ in $\xi_{i,1}$. Indeed this polynomial is:

$$P(t) = \prod_{\alpha=1}^{d}(R_1 + (A_{\alpha\alpha} - A_{11})t)^2 \sum_{\alpha=1}^{d}\left(\frac{R_\alpha t}{R_1 + (A_{\alpha\alpha} - A_{11})t}\right)^2 - \prod_{\alpha=1}^{d}(R_1 + (A_{\alpha\alpha} - A_{11})t)^2. \tag{4.3}$$

We observe that $P(t)$ is a polynomial of degree $2d$ and clearly, $\xi_{i,1}$ is a root for each $i = 1, \cdots, n$. Therefore, only $2d$ of $\xi_{i,1}$s can be distinct and in view of (4.2), only $2d$ of $\{\mathbf{x}_1^*, \cdots, \mathbf{x}_n^*\}$ can be distinct.

Definition 4.2.2 *Let $Q \in V_{n,d}$ and let $\mathbf{x}^*$ be a global minimum of Q on $\left(S^{d-1}\right)^n$. We call polynomial $P(t)$ of (4.3) the characteristic polynomial of the global minimum $(\mathbf{x}_1^*, \cdots, \mathbf{x}_n^*)$.*

Remark 4.2.3 *There are $\binom{n+2d-1}{n}$ distinct ways of populating a set $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ using $2d$*

distinct points on S^{d-1}. Therefore, this theorem reduces the search space of non-negativity of Q from an $n(d-1)$ dimensional manifold to $\binom{n+2d-1}{n}$ copies of a $2d(d-1)$ dimensional manifold.

We now address the question of whether the bound of Theorem 4.2.1 on the number of distinct coordinates is tight. Proving tightness of this bound can be accomplished by producing an example of a polynomial which attains a global minimum only at points with at least $2d$ of $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ distinct. It is straightforward to see that if there is such an example with $n > 2d$, then one can construct such an example with $n = 2d$. Therefore, it suffices to restrict ourselves to the case $n = 2d$ in the search for such polynomials. Lemma 4.2.4 below, provides a useful insight on zeroes of nonnegative polynomials with $n = 2d$.

Lemma 4.2.4 *A polynomial $Q \in V_{2d,d}$ has at most one critical point up to permutations with all of $\{\mathbf{x}_1^* \cdots, \mathbf{x}_{2d}^*\}$ distinct.*

Proof 4.2.2 *We will use the characteristic polynomial introduced in Definition 4.2.2. If $(\mathbf{x}_1^*, \cdots, \mathbf{x}_{2d}^*)$ is a global minimum with all of $\{\mathbf{x}_1^*, \cdots, \mathbf{x}_{2d}^*\}$ distinct, then the coordinates $\{\xi_{1,1}, \xi_{2,1}, \cdots, \xi_{2d,1}\}$ are the roots of (4.3). Therefore their sum, and the sum of their products taken two at a time are given by the second and third leading coefficients of the characteristic polynomial:*

$$\sum_{j=1}^{2d} \xi_{j,1} = -2R_1 \sum_{\alpha=2}^{2d} \frac{1}{A_{\alpha\alpha} - A_{11}} \implies \sum_{j=1}^{2d} \xi_{j,1} = \frac{-A_1 \sum_{\alpha=2}^{2d} \frac{1}{A_{\alpha\alpha} - A_{11}}}{\frac{1}{2} + A_{11} \sum_{\alpha=2}^{2d} \frac{1}{A_{\alpha\alpha} - A_{11}}} \tag{4.4}$$

The last implication follows from $R_1 = A_1 + A_{11} \sum_{j=1}^{2d} \xi_{j,1}$. In the characteristic polynomial, only R_α depends on the coordinates $\xi_{i,\alpha}$. However, the above equation shows that R_α depend only on Q, and are the same for each stationary point with all of $\{\mathbf{x}_1^*, \cdots, \mathbf{x}_{2d}^*\}$ distinct. Therefore, every such critical point has the same characteristic polynomial and therefore, there can be at most one critical point up to permutation with all of $\{\mathbf{x}_1^* \cdots, \mathbf{x}_{2d}^*\}$ distinct. We refer to this point as the fundamental critical point and the corresponding characteristic polynomial as the fundamental polynomial of Q and denote it by $P_0(t)$.

Note that while the fundamental stationary point may not always exist, the fundamental polynomial is always well defined. The former exists iff the all of the roots of the latter are real. In the light of this lemma, we are to search for values of the coefficients A_α and $A_{\alpha\alpha}$ for which the the fundamental stationary point is also the *unique* global minima. Theorem 4.2.5 below produces such an example for $d = 2$.

Theorem 4.2.5 *For $d = 2$, the bound of Theorem 4.2.1 is tight.*

Proof 4.2.3 *Let us define*

$$Q = A_0 + A_1 \sum_{j=1}^{4} x_{j,1} + A_2 \sum_{j=1}^{4} x_{j,2} + \left(1 - \frac{\Delta}{2}\right) \sum_{i<j} x_{i,1}x_{j,1} + \left(1 + \frac{\Delta}{2}\right) \sum_{i<j} x_{i,2}x_{j,2} \quad (4.5)$$

We show that for sufficiently small, positive values of Δ, the fundamental stationary point of Q is also its unique global minimum. The fundamental polynomial of Q is

$$P_0(t) = t^4 + t^3 A_1 + t^2 \left(\frac{A_1^2 + A_2^2}{4} - 1\right) - t A_1 - \frac{A_1^2}{4} \quad (4.6)$$

Using the fundamental polynomial, we can evaluate Q at its fundamental stationary point. We denote it by f_0:

$$f_0 = A_0 - \frac{A_1^2 + A_2^2}{2} - 2 \quad (4.7)$$

Note that this is independent of Δ. This means that this particular stationary value of Q is independent of Δ. We are to show that for sufficiently small positive values of Δ, every other critical value of Q is higher than f_0. Using $(x_i^1)^2 + (x_i^2)^2 = 1$, we may rewrite Q, in terms of $X = x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1}$ and $Y = x_{1,2} + x_{2,2} + x_{3,2} + x_{4,2}$ as:

$$Q = A_0 + \frac{1}{2}(X + A_1)^2 + \frac{1}{2}(Y + A_2)^2 - \frac{A_1^2 + A_2^2}{2} - 2 + \frac{\Delta}{2} \sum_{i<j} x_{i,2}x_{j,2} - \frac{\Delta}{2} \sum_{i<j} x_{i,1}x_{j,1} \quad (4.8)$$

Quite clearly, when $\Delta = 0$, f_0 is the global minima of Q and is attained at several stationary points, including the fundamental one. Let $\{\mathbf{x}_1^*(\Delta), \mathbf{x}_2^*(\Delta), \mathbf{x}_3^*(\Delta), \mathbf{x}_3^*(\Delta)\}$ be any

28

stationary point other than the fundamental one. Note that there can be only three distinct $\mathbf{x}_i^*(\Delta)s$. *The corresponding stationary value, using* $\mathbf{x}_i^*(\Delta) = (\xi_{i,1}(\Delta), \xi_{i,2}(\Delta))^T$, *is:*

$$
\begin{aligned}
f(\Delta) =& A_0 + \frac{1}{2}(\xi_{1,1}(\Delta) + \xi_{2,1}(\Delta) + 2\xi_{3,1}(\Delta) + A_1)^2 + \frac{1}{2}(\xi_{1,2}(\Delta) + \xi_{2,2}(\Delta) + 2\xi_{3,2}(\Delta) + A_2)^2 \\
&- \frac{A_1^2 + A_2^2}{2} - 2 + \frac{\Delta}{2}\sum_{i<j}\xi_{i,2}(\Delta)\xi_{j,2}(\Delta) - \frac{\Delta}{2}\sum_{i<j}\xi_{i,1}(\Delta)\xi_{j,1}(\Delta)
\end{aligned}
$$

$$(4.9)$$

If this is not a local minimum, then $f(0) > f_0$. *If it is a local minima,* $f(0) = f_0$ *and therefore it suffices to show that* $f'(0) > 0$ *(because* f_0 *is independent of* Δ). *It is straightforward to see that,*

$$
f'(0) = \frac{1}{2}\left(\sum_{i<j}\xi_{i,2}(0)\xi_{j,2}(0) - \sum_{i<j}\xi_{i,1}(0)\xi_{j,1}(0)\right) \tag{4.10}
$$

Here we have used the fact that at $\Delta = 0$, $\xi_{1,1} + \xi_{2,1} + 2\xi_{3,1} + A_1 = 0$ *and* $\xi_{1,2} + \xi_{2,2} + 2\xi_{3,2} + A_2 = 0$. *Hereafter, we use* $\xi_{i,\alpha}$ *instead of* $\xi_{i,\alpha}(0)$. *We compute* $f'(0)$ *by solving the equations:*

$$
\begin{aligned}
\xi_{1,1} + \xi_{2,1} &= A_1 - 2\xi_{3,1} \\
\xi_{1,2} + \xi_{2,2} &= A_2 - 2\xi_{3,2} \\
(\xi_{i,1})^2 + (\xi_{i,2})^2 &= 1
\end{aligned} \tag{4.11}
$$

We eliminate $\xi_{i,\alpha}$ *for* $i = 1, 2$ *and* $\alpha = 1, 2$ *to obtain the value of* $f'(0)$:

$$
f'(0) = (3(\xi_{3,1})^2 - 3(\xi_{3,2})^2 + 2\xi_{3,1}A_1 + 2\xi_{3,2}A_2)(A_1^2 + A_2^2 + 4 - 4\xi_{3,1}A_1 - 4\xi_{3,2}A_2) \tag{4.12}
$$

$\xi_3^{1,2}$ *are free parameters, constrained only by* $(\xi_{3,1})^2 + (\xi_{3,2})^2 = 1$ *and their range of values covers all local minima excluding the fundamental stationary point. It is quite easy to see that this quantity is strictly positive for any choice of* $\xi_{3,\alpha}$, *when* $0 < A_2 \ll A_1$.

CHAPTER 5

ASYMPTOTIC RESULTS

As noted in the previous chapter, while the dimension of the test manifold is independent of n, the number of copies of it scales as n^d, leading to the number of inequality criteria that check for membership of a point in $C^*_{n,d}$ also scaling as n^d. However, quite often, one can find a convex cone characterized by a finite number of inequality criteria (independent of n) that asymptotically approximates $C^*_{n,d}$, leading to an asymptotically tight criteria to determine membership of a point in $C^*_{n,d}$. In this chapter, we develop such asymptotic criteria. Before we begin, we illustrate this idea with a simple example.

Let $\mathcal{P}_n$ be the cone with an n-sided regular polygon as the cross section, defined as:

$$\mathcal{P}_n = \text{conic.hull}\left\{\left(\cos\left(k\frac{2\pi}{n}\right), \sin\left(k\frac{2\pi}{n}\right), 1\right) : \ k = 0, 1, \cdots, n-1\right\} \qquad (5.1)$$

Figure 5.1 shows a cross section of this cone. A point (x, y, z) is in $\mathcal{P}_n$ if and only if it satisfies n linear inequalities:

$$
\begin{aligned}
y - m_k x - c_k &\leq 0; \ k = 0, 1, \cdots, n-1 \\
m_k &= \frac{\sin(2\pi(k+1)/n) - \sin(2\pi k/n)}{\cos(2\pi(k+1)/n) - \cos(2\pi k/n)} \\
c_k &= \frac{\sin(2\pi/n)}{\cos(2\pi(k+1)/n) - \cos(2\pi k/n)}
\end{aligned}
\qquad (5.2)
$$

Although it is apparent that the number of inequalities grow linearly with n, one can find an asymptotically tight criterion whose complexity is independent of n. Note that the circumcircle of the polygon defines a cone that contains $\mathcal{P}_n$: $\mathcal{C} = \{(x, y, z) : \ z^2 - x^2 - y^2 \geq 0\} \supseteq \mathcal{P}_n$ and therefore provides a necessary condition for inclusion in $\mathcal{P}_n$. Moreover, the incircle of the polygon defines a cone inside $\mathcal{P}_n$, i.e., $\{(x, y, z) : \ \cos^2(\pi/n)z^2 - x^2 - y^2 \geq 0\} \subseteq$
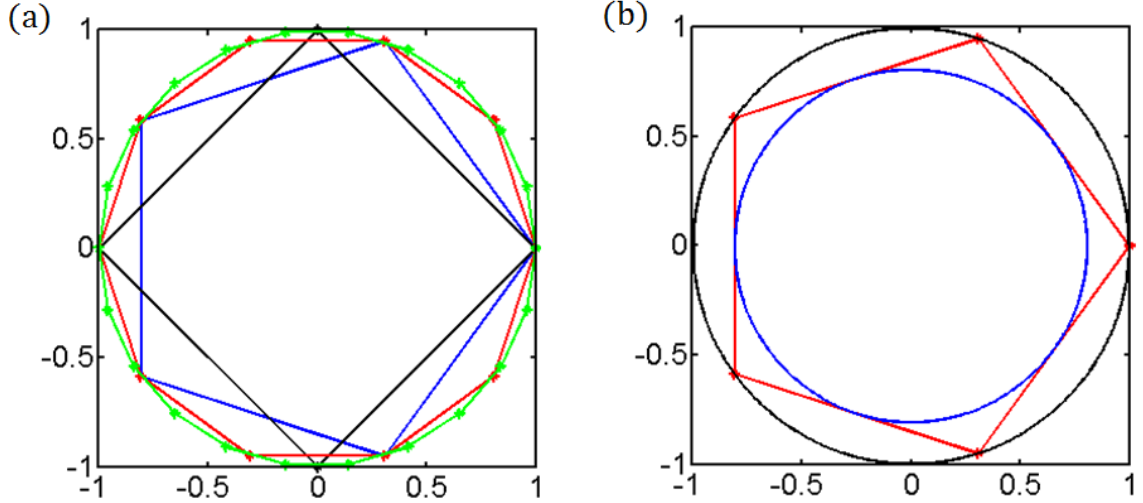
Figure 5.1: (a) shows the cross sections of cones $\mathcal{P}_n$ for $n = 4$ (green), $n = 5$ (blue), $n = 10$ (red) and $n = 22$ (red). (b) shows the cross section of $\mathcal{P}_5$ (red), its incircle (blue) and circumcircle (black).

$\mathcal{P}_n$ and provides a sufficient criterion. In other words $(x, y, z) \in \mathcal{P}_n$ if $\cos^2(\pi/n)z^2 - x^2 - y^2 \geq 0$ and only if $z^2 - x^2 - y^2 \geq 1$. Note that these two conditions approach each other in the limit of large n. In other words, the necessary condition is asymptotically sufficient and the sufficient condition is asymptotically necessary.

In this chapter, we identify a cone $G^*_{n,d}$, characterized by a fixed number of inequalities that asymptotically approximates $C^*_{n,d}$, in the sense that $G^*_{n,d} \subseteq C^*_{n,d}$ and when scaled up by a small amount it includes $C^*_{n,d}$. That is, for some $\epsilon > 0$, $(1 + \epsilon)G^*_{n,d} \supseteq C^*_{n,d}$ and $\epsilon \to 0$ as $n \to \infty$. In this sense, we obtain an asymptotically tight criterion for membership in $C^*_{n,d}$, which we use to develop asymptotically tight criterion for membership in $C_{n,d}$.

5.1 The Limiting Cone

The cones $C^*_{n,d}$ are nested, i.e., $C^*_{n,d} \subset C^*_{n-1,d}$. This follows from the simple observation that any polynomial $P \in C^*_{n-1,d}$ can be written as

$$P(\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_{n-1}) = Q(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n-1}, \mathbf{0}) \tag{5.3}$$

with $Q \in C^*_{n,d}$. Since the cones $C^*_{n,d}$ are nested, the limiting cone is expected to be their intersection: $\cap_n C^*_{n,d}$. However, this intersection is rather small, as shown in Figure 5.2. The lemma below states it more precisely:

Lemma 5.1.1 *The limit cone, $\cap_n C^*_{n,d}$ is the cone of all constant polynomials:*

$$\cap_n C^*_{n,d} = \{A_0 : \ A_0 \geq 0\}.$$

Proof 5.1.1 *Consider a polynomial $Q = (A_0, A_1, \cdots A_n, A_{11}, \cdots A_{dd}) \in \cap_n C^*_{n,d}$. We can show that Q the only non-zero coefficient of Q is A_0 by evaluating Q on a simple collection of points.*

Let n be any positive integer. By definition, $Q \in C^*_{2n,d}$. Let $\mathbf{x}(k) = (\mathbf{x}_1(k), \cdots \mathbf{x}_{2n}(k))$ be a point in K^d_n defined as:

$$\mathbf{x}_i(k) = \begin{cases} (1, 0, \ldots, 0) \in D^d \text{ when } i = 1, 2 \cdots k \\ \\ (-1, 0, \ldots, 0) \in D^d \text{ when } i = k+1, k+2, \ldots, 2n \end{cases}$$

where, $0 \leq k \leq 2n$ is some integer. The evaluation of Q at $\mathbf{v}(k)$ is given by:

$$Q(\mathbf{x}_1, \cdots \mathbf{x}_{2n}) = A_0 + A_1(2k - 2n) + A_{11}(2(k-n)^2 - n)$$

When $k = n$, we have $Q(\mathbf{v}(n)) = A_0 - nA_{11} \geq 0$ for all n. It follows that $A_{11} = 0$. Further, when $k = 0$, we have $Q(\mathbf{x}(0)) = A_0 - 2nA_1 \geq 0$. Therefore, $A_1 = 0$. Using similar arguments, it follows that $A_\alpha = A_{\alpha\alpha} = 0$ for $\alpha = 1, 2, \ldots, d$. Thus, $\cap_n C^*_{n,d}$ is the cone of constant polynomials.

In the following, we show that after a suitable rescaling of the cones, the limiting cone

is a non-trivial semi algebraic set. Let us define the rescaled cones as:

$$\tilde{C}^*_{n,d} = \left\{(A_0, \sqrt{n}A_1, \sqrt{n}A_2, \cdots \sqrt{n}A_d, nA_{11}, nA_{22}, \cdots nA_{dd})^T : (A_0, A_1, \cdots A_n, A_{11}, \cdots A_{dd})^T \in C^*_{n,d}\right\} \tag{5.4}$$

That is, we rescale the linear coefficients by $\sqrt{n}$ and the quadratic coefficients by n. The rescaled cones are no longer nested. However, their intersection, $\cap_n \tilde{C}^*_n$, is non-trivial is it is a limit of the the cones. We begin with the simple case of $d = 1$.

5.1.1 The One Dimensional Case d=1

Membership of a quadratic $Q = (A_0, A_1, A_{11})$ in $C^*_{n,1}$ can be checked by the $n + 1$ inequalities in proposition 5.1.1. That is, $(A_0, A_1, A_{11}) \in C^*_{n,1}$ if and only if $A_0 + A_1(n - 2k) + A_{11}((n - 2k)^2 - n) \geq 0$ hold for $k = 0, 1, \cdots, n$. Let $B_0 = A_0$, $B_1 = \sqrt{n}A_1$ and $B_{11} = nA_{11}$ be the rescaled coefficients. It follows that $Q = (B_0, B_1, B_{11}) \in \tilde{C}^*_{n,1}$ iff

$$B_0 - B_{11} + B_1\left(\frac{n - 2k}{\sqrt{n}}\right) + B_{11}\left(\frac{n - 2k}{\sqrt{n}}\right)^2 \geq 0 \tag{5.5}$$

holds for $k = 0, 1, \cdots, n$. Introducing $X = \left(\frac{n-2k}{\sqrt{n}}\right)$ and polynomial $P(X) = B_0 - B_{11} + B_1 X + B_{11} X^2$, we may re-write the above conditions as $P(X) \geq 0$ for $X = -\sqrt{n}, -\sqrt{n} + 2/\sqrt{n}, \cdots, +\sqrt{n}$. In other words, $Q \in C^*_{n,1}$ iff $P(X) \geq 0$ on $n + 1$ evenly spaced points in $[-\sqrt{n}, \sqrt{n}]$. Given that the spacing $2/\sqrt{n}$ approaches zero an n approaches infinity, we are prompted to define the following cone:

$$\tilde{G}^*_{n,1} = \left\{(B_0, B_1, B_{11}) : P(X) \geq 0 \ \forall \ X \in [-\sqrt{n}, \sqrt{n}]\right\} \tag{5.6}$$

Clearly, the sets $\tilde{G}^*_{n,1}$ are nested, i.,e., $\tilde{G}^*_{n,1} \subseteq \tilde{C}^*_{n,1}$. Moreover, $\tilde{G}^*_{n,1} \supseteq \tilde{G}^*_{n+1,1}$ and $\cap_n \tilde{G}^*_{n,1} = \tilde{G}^*$ is the space of all non-negative polynomials $P(X)$ on $\mathbb{R}$.

Unlike $\tilde{C}^*_{n,1}$, $\tilde{G}^*$ and $\tilde{G}^*_{n,1}$ do not need n inequalities. In particular, $\tilde{G}^*$ is defined one
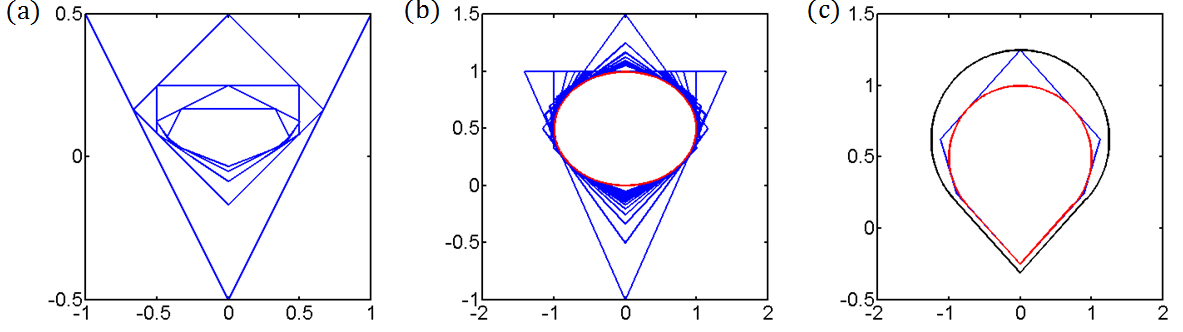
Figure 5.2: (a) shows the cross sections of $C_{n,1}^*$ for $n = 2$ to $n = 6$. (b) shows the cross sections of the rescaled cones $\tilde{C}_{n,1}^*$ for $n = 2$ to $n = 20$ in blue and the cross section of the limiting cone $\tilde{G}^*$ in red. (c) shows $\tilde{C}_{5,1}^*$ in blue and the corresponding approximation, $\tilde{G}_{5,1}^*$ in red. The expanded cone, $\tilde{G}_{5,1,\epsilon}^*$, for $\epsilon = 1/4$ is shown in black.

inequality. $(B_0, B_1, B_{11}) \in \tilde{G}^*$ iff

$$B_0^2 - B_1^2 - (2B_{11} - 1)^2 \geq 0 \tag{5.7}$$

This defines an ellipse, shown in red in Figure 5.2(b). This already provides a necessary condition for inclusion in $\tilde{C}_{n,1}^*$. It can be seen from Figure 5.2(b) that this condition is asymptotically sufficient. Using $\tilde{G}_{n,1}^*$, moreover, one can find stronger conditions for inclusion in $\tilde{C}_{n,1}^*$. The former includes polynomials of three kinds

(i) $B_{11} > 0$, both zeros of the polynomial $P(X)$ greater than $\sqrt{n}$, or both lesser than $-\sqrt{n}$.

(ii) $B_{11} > 0$, both zeros of the polynomial $P(X)$ complex (i.e., not real).

(iii) $B_{11} < 0$, one of the zeros of $P(X)$ less than $-\sqrt{n}$ and the other greater than $+\sqrt{n}$.

It is straightforward to check for inclusion in $\tilde{G}_{n,1}^*$, based on the above ideas. This also involves a fixed number of inequalities. The cross section of $\tilde{G}_{n,1}^*$ can be understood by considering its extreme points. Four classes of extreme points may be easily identified:

(i) $B_{11} > 0$, one zero equal to $\sqrt{n}$ and the other greater than $\sqrt{n}$. That is, $P(X) = B_{11}(X - \sqrt{n})(X - \alpha)$ for $\alpha \geq \sqrt{n}$.

34

(ii) $B_{11} > 0$, one zero equal to $-\sqrt{n}$ and the other less than $-\sqrt{n}$. That is, $P(X) = B_{11}(X + \sqrt{n})(X + \alpha)$ for $\alpha \geq \sqrt{n}$.

(iii) $B_{11} < 0$, two zeros at $\pm\sqrt{n}$ respectively. That is, $P(X) = B_{11}(X^2 - n)$.

(iv) $B_{11} > 0$, a common root at $\alpha \in [-\sqrt{n}, \sqrt{n}]$. That is, $P(X) = B_{11}(X - \alpha)^2$, $\alpha \in [-\sqrt{n}, \sqrt{n}]$.

Based on the above set of extreme points, it is straightforward to construct the cross section of $\tilde{G}^*_{n,1}$, as shown in Figure 5.2(c) in red. To show that the necessary conditions obtained by inclusion in $\tilde{G}^*_{n,1}$ are asymptotically sufficient, we consider the following cone:

$$\tilde{G}^*_{n,1,\epsilon} = \left\{ (B_0, B_1, B_{11}) : \; ((1 + \epsilon)B_0, B_1, B_{11}) \in \tilde{G}^*_{n,1} \right\} \tag{5.8}$$

This cone is obtained by slightly expanding $\tilde{G}^*_{n,1}$. We show, in the below lemma, that when $\epsilon \geq 1/(n-1)$, this expanded cone contains $\tilde{C}^*_{n,1}$ and therefore provides a sufficient condition. In the limit of large n, the necessary and sufficient conditions converge.

Proposition 5.1.2 *For $\epsilon \geq \frac{1}{n-1}$, $\tilde{G}^*_{n,1} \subseteq \tilde{C}^*_{n,1} \subseteq \tilde{G}^*_{n,1,\epsilon}$*

Proof 5.1.2 *We show that all the extreme points in $\tilde{C}^*_{n,1}$ are in $\tilde{G}^*_{n,1,\epsilon}$. Let $(B_0, B_1, B_{11}) \in \tilde{C}^*_{n,1}$ be an extreme point. It follows that the corresponding polynomial $P(X)$ takes a zero value at two consecutive points in the set $\{-\sqrt{n}, -\sqrt{n} + 2/\sqrt{n}, \cdots, +\sqrt{n}\}$. The zeros of P are therefore separated by $2/\sqrt{n}$. Consequently, the minimum value of P is $-B_{11}/n$. It follows that $P + B_{11}/n$ is non-negative on $[-\sqrt{n}, \sqrt{n}]$ and so is $P + B_0/(n-1)$. Thus, $P \in \tilde{G}^*_{n,1,\epsilon}$.*

The dual cones of $\tilde{G}^*_{n,1}$ and $\tilde{G}^*_{n,1,\epsilon}$, which we refer to as $\tilde{G}_{n,1}$ and $\tilde{G}_{n,1,\epsilon}$ respectively satisfy

$$\tilde{G}_{n,1,\epsilon} \subseteq \tilde{C}_{n,1} \subseteq \tilde{G}_{n,1} \tag{5.9}$$

providing necessary and sufficient criteria for membership in the moment cone $C_{n,1}$. We now proceed to define and characterize $\tilde{G}^*_{n,d}$ for arbitrary d.

5.1.2 General Dimension

We proceed along similar lines as for the $d = 1$ case to obtain a necessary and asymptotically sufficient condition for membership in $\tilde{C}^{*}_{n,d}$. Following the intuition gained in the $d = 1$ case, we rewrite the quadratic form $Q = (A_0, A_\alpha, A_{\alpha\alpha})$ in the scaled variables $B_0 = A_0$, $B_\alpha = \sqrt{n}A_\alpha$ and $B_{\alpha\alpha} = nA_{\alpha\alpha}$:

$$Q(\mathbf{x}) = P(X,Y) = B_0 + \sum_{\alpha=1}^{d} B_\alpha X_\alpha + \frac{n-1}{n} \sum_{\alpha=1}^{d} B_{\alpha\alpha} X_\alpha^2 - \sum_{\alpha=1}^{d} B_{\alpha\alpha} Y_\alpha^2 \qquad (5.10)$$

where the variables X_α and Y_α are defined as

$$X_\alpha = \sum_{i=1}^{n} \frac{x_{i,\alpha}}{\sqrt{n}}$$

$$Y_\alpha = \sqrt{\sum_{i=1}^{n} \frac{x_{i,\alpha}^2}{n} - \frac{X_\alpha^2}{n}} \qquad (5.11)$$

defined for $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$ and $\mathbf{x}_i = (x_{i,1}, \cdots, x_{i,d})$. Note that the Cauchy-Schwartz inequality ensures that Y_α are well defined real numbers. The variables X_α and Y_α satisfy $\sum_\alpha Y_\alpha^2 + X_\alpha^2/n = ||Y||^2 + \frac{1}{n}||X||^2 \leq 1$ and this follows from the conditions $||\mathbf{x}_i|| \leq 1$. Although the above mentioned conditions on X and Y are weaker than what is implied from $||\mathbf{x}_i|| \leq 1$, taking a cue from the case of $d = 1$, we define the cone $\tilde{G}^{*}_{n,d}$:

$$\tilde{G}^{*}_{n,d} = \left\{ (B_0, B_\alpha, B_{\alpha\alpha}) : \ P(X,Y) \geq 0 \ \forall \ X, Y \ \text{s.t} \ ||Y||^2 + \frac{1}{n}||X||^2 \leq 1 \right\} \qquad (5.12)$$

Clearly, $\tilde{G}^{*}_{n,d} \subseteq \tilde{C}^{*}_{n,d}$, providing a necessary condition for membership in the latter. This necessary condition can be expressed as a linear matrix inequality using the S-lemma. We return to the exact criteria in the next chapter. Similar to the $d = 1$ case, it is straightforward to see that $\tilde{G}^{*}_{n+1,d} \subseteq \tilde{G}^{*}_{n,d}$ and therefore, we define $\tilde{G}^{*}_{d} = \cup_n \tilde{G}^{*}_{n,d}$, that also provides an albeit weaker, necessary condition.

In the following, we prove that the necessary condition provided by $\tilde{G}^{*}_{n,d}$ is asympto-

tically sufficient, following the same line of arguments as before. For $\epsilon > 0$, we define $\tilde{G}^*_{n,d,\epsilon}$, by expanding $\tilde{G}^*_{n,d}$:

$$\tilde{G}^*_{n,d,\epsilon} = \{(B_0, B_\alpha, B_{\alpha\alpha}) : \ ((1+\epsilon)B_0, B_\alpha, B_{\alpha\alpha}) \in \tilde{G}^*_{n,d}\} \tag{5.13}$$

We show that for $\epsilon \geq \frac{1}{n-1}$, $\tilde{C}^*_{n,d} \subseteq \tilde{G}^*_{n,d,\epsilon}$ and therefore it provides a sufficient condition. Furthermore, the necessary and sufficient conditions asymptotically approach each other, showing that the former is asymptotically sufficient. Theorem 6.1.4 is the main result of this chapter and it is a direct generalization of proposition 6.1.2. To prove this theorem, we need a technical lemma, which we prove at the end of this chapter:

Lemma 5.1.3 *Let* $n \in \mathbb{N}$ *and* $X = (X_1, \cdots, X_d) \in \mathbb{R}^d$ *such that* $||X||^2 = X_1^2 + \cdots + X_d^2 \leq n$. *There exists* $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n) \in K_n^d$ *with* $\mathbf{x}_i = (x_{i,1}, \cdots, x_{i,d})$ *such that:*

$$\sum_{i=1}^n \frac{x_{i,\alpha}}{\sqrt{n}} = X_\alpha \, for \, \alpha = 1, 2, \cdots, d$$

$$\sum_{i=1}^n \frac{x_{i,\alpha}^2}{n} - \frac{X_\alpha^2}{n} = 0 \, for \, \alpha = 1, 2, \cdots, d-1 \tag{5.14}$$

$$\left| \left(\sum_{i=1}^n \frac{x_{i,d}^2}{n} - \frac{X_d^2}{n} \right) - \left(1 - \frac{||X||^2}{n} \right) \right| \leq \frac{1}{n}$$

We defer the proof of this lemma to the end of this chapter. We are now ready to state and prove theorem 6.1.4.

Theorem 5.1.4 *For* $\epsilon \geq \frac{1}{n-1}$, *it follows that*

$$\tilde{G}^*_{n,d} \subseteq \tilde{C}^*_{n,d} \subseteq \tilde{G}^*_{n,d,\epsilon} \tag{5.15}$$

Proof 5.1.3 *The first inclusion follows trivially from the definition of* $\tilde{G}^*_{n,d}$. *To show the second inclusion, let* $Q = (B_0, B_\alpha, B_{\alpha\alpha}) \in \tilde{C}^*_{n,d}$. *Recall that the polynomial* $P(X,Y)$ *was*

defined as

$$P(X,Y) = B_0 + \sum_{\alpha=1}^{d} B_\alpha X_\alpha + \frac{n-1}{n} \sum_{\alpha=1}^{d} B_{\alpha\alpha} X_\alpha^2 - \sum_{\alpha=1}^{d} B_{\alpha\alpha} Y_\alpha^2 \qquad (5.16)$$

In order to show that $P + \epsilon B_0 \geq 0$ whenever $||Y||^2 + \frac{1}{n}||X||^2 \leq 1$, we pick a point (X,Y) that satisfies the latter condition and approximate it using $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$ with the help of lemma 5.1.3.

If $B_{\alpha\alpha} \leq 0$ for $\alpha = 1, 2, \cdots, d$, we choose $\mathbf{x}_i = \frac{X}{\sqrt{n}}$. The condition $||Y||^2 + \frac{1}{n}||X||^2 \leq 1$ ensures that $\mathbf{x} \in K_n^d$. It follows now that

$$P(X,Y) \geq B_0 + \sum_{\alpha=1}^{d} B_\alpha X_\alpha + \frac{n-1}{n} \sum_{\alpha=1}^{d} B_{\alpha\alpha} X_\alpha^2 = Q(\mathbf{x}) \geq 0 \qquad (5.17)$$

The last inequality follows from $Q \in \tilde{C}_{n,d}^*$. If $B_{\alpha\alpha} > 0$ for atleast one α, we assume without loss of generality that $B_{dd} \geq B_{\alpha\alpha}$ for $\alpha = 1, 2, \cdots d - 1$. It also follows that $B_{dd} > 0$. Clearly,

$$P(X,Y) \geq B_0 + \sum_{\alpha=1}^{d} B_\alpha X_\alpha + \frac{n-1}{n} \sum_{\alpha=1}^{d} B_{\alpha\alpha} X_\alpha^2 - B_{dd}\left(1 - \frac{||X||^2}{n}\right) \qquad (5.18)$$

We now use lemma 5.1.3 to pick $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$ such that

$$X'_\alpha = \sum_{i=1}^{n} \frac{x_{i,\alpha}}{\sqrt{n}} = X_\alpha \text{ for } \alpha = 1, 2, \cdots, d$$

$$Y'^2_\alpha = \sum_{i=1}^{n} \frac{x_{i,\alpha}^2}{n} - \frac{X_\alpha^2}{n} = 0 \text{ for } \alpha = 1, 2, \cdots, d-1 \qquad (5.19)$$

$$\left| Y'^2_d - \left(1 - \frac{||X||^2}{n}\right) \right| = \left| \left(\sum_{i=1}^{n} \frac{x_{i,d}^2}{n} - \frac{X_d^2}{n} \right) - \left(1 - \frac{||X||^2}{n}\right) \right| \leq \frac{1}{n}$$

The above equations enable us to evaluate $Q(\mathbf{x})$ and we thus obtain

$$Q(\mathbf{x}) = B_0 + \sum_{\alpha=1}^{d} B_\alpha X_\alpha + \frac{n-1}{n} \sum_{\alpha=1}^{d} B_{\alpha\alpha} X_\alpha^2 - B_{dd} Y_d'^2 \tag{5.20}$$

Finally using equation 5.18, we obtain

$$P(X,Y) + \frac{B_{dd}}{n} \geq Q(\mathbf{x}) \geq 0 \tag{5.21}$$

It follows from the $d = 1$ case that $B_{dd} \leq \frac{n}{n-1} B_0$ and therefore, $P + \epsilon B_0 \geq 0$ whenever $\epsilon \geq \frac{1}{n-1}$.

Proof 5.1.4 (Proof of Lemma 5.1.3) *We construct a point $\mathbf{x} \in K_n^d$ with the claimed properties. Let*

$$\mathbf{x}_i = \begin{cases} \left(\frac{X_1}{\sqrt{n}}, \frac{X_2}{\sqrt{n}}, \cdots \frac{X_{d-1}}{\sqrt{n}}, \sqrt{1 - \sum_{\alpha=1}^{d-1} \left(\frac{X_\alpha}{\sqrt{n}} \right)^2} \right) & \text{for } i = 1, 2, \cdots k \\ \left(\frac{X_1}{\sqrt{n}}, \frac{X_2}{\sqrt{n}}, \cdots \frac{X_{d-1}}{\sqrt{n}}, -\sqrt{1 - \sum_{\alpha=1}^{d-1} \left(\frac{X_\alpha}{\sqrt{n}} \right)^2} \right) & \text{for } i = k+1, k+2, \cdots n-1 \\ \left(\frac{X_1}{\sqrt{n}}, \frac{X_2}{\sqrt{n}}, \cdots \frac{X_{d-1}}{\sqrt{n}}, z \right) & \text{for } i = n \end{cases} \tag{5.22}$$

We make an appropriate choice of k and z in the following. It follows that

$$\sqrt{1 - \sum_{\alpha=1}^{d-1} \left(\frac{X_\alpha}{\sqrt{n}} \right)^2} \geq \frac{|X_d|}{\sqrt{n}} \tag{5.23}$$

Each $x_{i,d}$ is equal to $\pm\sqrt{1 - \sum_{\alpha=1}^{d-1} \left(\frac{X_\alpha}{\sqrt{n}} \right)^2}$ and k is the number of them with a $+$ sign. We may choose k in such a way that the sum of all the $x_{i,d}$'s is closest to $\sqrt{n} X^d$. That is,

$$\left| \sqrt{n} X_d - \sum_{i=1}^{n-1} x_{i,d} \right| \leq \sqrt{1 - \sum_{\alpha=1}^{d-1} \left(\frac{X_\alpha}{\sqrt{n}} \right)^2} \leq 1 \tag{5.24}$$

Let us now choose $z = \left(\sqrt{n}X_d - \sum_{i=1}^{n-1} x_{i,d}\right)$. It follows from the above inequality that this is a valid choice for a point on K_n^d. It also follows that $\frac{\sum_i x_i^\alpha}{\sqrt{n}} = X^\alpha$ for $\alpha = 1, 2, \cdots, d$. It remains to show the last inequality in the lemma. Explicitly,

$$\frac{\sum_i x_{i,d}^2}{n} - \left(\sum_i \frac{x_{i,d}}{n}\right)^2 = \left(1 - \frac{1}{n}||X||^2\right) - \frac{1}{n}\left(1 - ||\mathbf{x}_n||^2\right) \tag{5.25}$$

5.2 Necessary and sufficient criteria for moments

In this section, using the cones $\tilde{G}_{n,d}^*$ and $\tilde{G}_{n,d,\epsilon}^*$, we develop a necessary condition and a sufficient condition for membership of a vector $(y_0, y_1, \cdots, y_d, y_{11}, y_{22}, \cdots, y_{dd})$ in the moment cone $\mathbb{C}_{n,d}$. We also show that these two conditions approach each other, i.e., the necessary condition is asymptotically sufficient and the sufficient condition is asymptotically necessary. Let us define the cones $\tilde{C}_{n,d}$ by rescaling $C_{n,d}$:

$$\tilde{C}_{n,d} = \left\{z_0, z_1, \cdots, z_d, z_{11}, \cdots, z_{dd} : \ (z_0, \sqrt{n}z_1, \cdots, \sqrt{n}z_d, nz_{11}, \cdots, nz_{dd}) \in C_{n,d}\right\}$$

Let us also define $\tilde{G}_{n,d,\epsilon}$ as the dual of $\tilde{G}_{n,d,\epsilon}^*$. It follows from Theorem 5.1.4 that

$$\tilde{G}_{n,d} \supseteq \tilde{C}_{n,d} \supseteq \tilde{G}_{n,d,\epsilon} \ \text{ for } \epsilon \geq \frac{1}{n-1} \tag{5.26}$$

Thus, membership in $\tilde{G}_{n,d}$ is a necessary condition and membership in $\tilde{G}_{n,d,\epsilon}$ is a sufficient condition for membership in $\tilde{C}_{n,d}$. In the following we develop inequality criteria to check for membership in $\tilde{G}_{n,d,\epsilon}$ and $\tilde{G}_{n,d}$, expressed as Linear Matrix Inequalities (LMI).

5.2.1 Characterizing the cone $\tilde{G}_{n,d,\epsilon}^*$

Note that $\tilde{G}_{n,d} = \tilde{G}_{n,d,\epsilon=0}$. Therefore, we begin with a characterization of $\tilde{G}_{n,d,\epsilon}^*$. By definition,

$$\tilde{G}_{n,d,\epsilon}^* = \left\{ (B_0, B_\alpha, B_{\alpha\alpha}) : \ P_\epsilon(X,Y) \geq 0 \ \forall \ X, Y \ \text{s.t} \ ||Y||^2 + \frac{1}{n}||X||^2 \leq 1 \right\} \quad (5.27)$$

Where the polynomial P_ϵ is defined as $P_\epsilon(X,Y) = (1+\epsilon)B_0 + \sum_\alpha B_\alpha X_\alpha + \frac{n-1}{n} \sum_\alpha B_{\alpha\alpha}(X_\alpha)^2 - \sum_\alpha B_{\alpha\alpha}(Y_\alpha)^2$. It can be expressed in terms of a $2d+1 \times 2d+1$ matrix $\mathbf{B}'$ and a $2d+1$ dimensional vector $\mathbf{X}'$ as $P = \mathbf{X}'^T \mathbf{B}' \mathbf{X}'$ where,

$$\mathbf{B}' = \begin{pmatrix} (1+\epsilon)B_0 & \frac{B_1}{2} & \cdots & \frac{B_d}{2} & 0 & 0 & \cdots & 0 \\ \frac{B_1}{2} & \frac{n-1}{n}B_{11} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & 0 & 0 & \cdots & 0 \\ \frac{B_d}{2} & 0 & \cdots & \frac{n-1}{n}B_{dd} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -B_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & -B_{22} & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -B_{dd} \end{pmatrix} \quad (5.28)$$

And

$$
\mathbf{X}' = \begin{pmatrix} 1 \\ X_1 \\ X_2 \\ \vdots \\ X_d \\ Y_1 \\ Y_2 \\ \vdots \\ Y_d \end{pmatrix} \tag{5.29}
$$

The constraint $||Y||^2 + \frac{1}{n}||X||^2 \leq 1$ can also be written in terms of a matrix $\mathbf{X}'^T \mathbf{M}' \mathbf{X}' \leq 0$, where

$$
\mathbf{M}' = \begin{pmatrix}
n & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -n & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & -n & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & 0 & \ddots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -n
\end{pmatrix} \tag{5.30}
$$

We may now define $\tilde{G}_{n,d,\epsilon}$ as $\{\mathbf{B}' : \mathbf{X}'^T \mathbf{B}' \mathbf{X}' \geq 0 \text{ whenever } \mathbf{X}'^T \mathbf{M}' \mathbf{X}' \leq 0\}$. By the S-lemma, it follows that $\mathbf{B}' \in \tilde{G}_{n,d,\epsilon}$ iff $\exists \lambda \geq 0$ such that $\mathbf{B}' + \lambda \mathbf{M}' \succeq 0$. The inconvenient condition that that $\lambda \geq 0$ can be eliminated by the following trick. Let us define $\mathbf{M}$ and $\mathbf{B}$ as

$$
\mathbf{B} = \begin{pmatrix} \mathbf{B}' & 0 \\ 0 & 0 \end{pmatrix} ; \quad \mathbf{M} = \begin{pmatrix} \mathbf{M}' & 0 \\ 0 & 1 \end{pmatrix} \tag{5.31}
$$

It now follows that $\mathbf{B} \in \tilde{G}_{n,d,\epsilon}$ iff $\exists \lambda \in \mathbb{R}$ such that $\mathbf{B} + \lambda \mathbf{M} \succeq 0$. To express this characterization in a more concise way, let us define the space of relevant positive semidefinite $2d + 2 \times 2d + 2$ matrices, $\mathcal{V}_+$

$$
\mathcal{V}_+ = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : A_1, A_2 \in S^{d+1} \right\}
\tag{5.32}
$$

Where S^{d+1} is the PSD cone of $d + 1 \times d + 1$ matrices. It is straightforward to see that $\mathcal{V}_+$ is self dual. The condition $\mathbf{B} + \lambda \mathbf{M} \succeq 0$ implies that $\mathbf{B} \in \mathcal{V}_+ \oplus \mathrm{span}(\mathbf{M})$. However, $\mathbf{B}$ is *any* $2d + 2 \times 2d + 2$ matrix. It belongs to a $2d + 1$ dimensional subspace spanned by $E_0, E_1, \cdots E_d, E_{11}, \cdots, E_{dd}$ defined as

$$
E_0(\epsilon) = \begin{pmatrix} 1+\epsilon & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} ; \quad E_\alpha = \begin{pmatrix} 0 & \cdots & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & 0 & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}
\tag{5.33}
$$

And

$$
E_{\alpha\alpha} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{n-1}{n} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}
\tag{5.34}
$$

We may now write $\mathbf{B} = B_0 E_0(\epsilon) + B_1 E_1 + \cdots B_d E_d + B_{11} E_{11} + \cdots + B_{dd} E_{dd}$. Let $\mathcal{U} = \text{span}(\mathbf{M})$ and $\mathcal{L} = \text{span}\{E_0(1 + \epsilon), E_\alpha, E_{\alpha\alpha}\}$. It is now straightforward to show that

$$\tilde{G}^*_{n,d,\epsilon} = \Pi_{\mathcal{L}} \left((\mathcal{U} \oplus \mathcal{L}) \cap \mathcal{V}_+\right) \tag{5.35}$$

Where $\Pi_{\mathcal{L}}$ is the projection to the subspace $\mathcal{L}$. The projection can be re-written, in terms of the orthogonal complement $\mathcal{L}^*$ of $\mathcal{L}$ as

$$\tilde{G}^*_{n,d,\epsilon} = \mathcal{L} \cap \left((\mathcal{U} \oplus \mathcal{L}) \cap \mathcal{V}_+ \oplus \mathcal{L}^*\right) \tag{5.36}$$

The inclusion in $\tilde{G}^*_{n,d,\epsilon}$ is straightforward to check via an LMI, i.e., $\mathbf{B} \in \tilde{G}^*_{n,d,\epsilon}$ iff $(\mathbf{B} \oplus \mathcal{U} \cap \mathcal{V}_+) \neq \emptyset$. In the next section, we characterize the dual of $\tilde{G}^*_{n,d,\epsilon}$ and develop necessary and sufficient conditions for inclusion of a given point, in $C_{n,d}$.

5.2.2 The dual of $\tilde{G}^*_{n,d,\epsilon}$

It now follows, from Eq. 5.36 that the dual, $\tilde{G}_{n,d,\epsilon}$ is given by

$$\tilde{G}_{n,d,\epsilon} = \mathcal{L}^* \oplus \mathcal{L} \cap \left((\mathcal{L}^* \cap \mathcal{U}^*) \oplus \mathcal{V}_+\right) \tag{5.37}$$

This provides a criterion for membership of a vector $(y_0, y_1, \cdots, y_d, y_{11}, y_{22}, \cdots, y_{dd})$ in $C_{n,d}$. It is equivalent to check for membership of the rescaled vector, $(y_0, y_1/\sqrt{n}, \cdots, y_d/\sqrt{n}, y_{11}/n, y_{22}/n$ in $\tilde{C}_{n,d}$. Corresponding to the basis elements $E_0(\epsilon), E_\alpha, E_{\alpha\alpha}$ of $\mathcal{L}$, we define a dual set of basis $\{F_0, F_\alpha, F_{\alpha\alpha}\}$ satisfying $\text{Tr}(E_i F_j) = \delta_{ij}$. It is straightforward to see that

$$F_0(\epsilon) = \frac{1}{(1 + \epsilon)^2} E_0(\epsilon)$$

$$F_\alpha = 2 E_\alpha \tag{5.38}$$

$$F_{\alpha\alpha} = \frac{n^2}{2n^2 - 2n + 1} E_{\alpha\alpha}$$

44

Corresponding to a vector $(y_0, y_1, \cdots, y_d, y_{11}, y_{22}, \cdots, y_{dd})$ we may define a $2d+2 \times 2d+2$ matrix

$$\mathbf{Y}_\epsilon = y_0 F_0(\epsilon) + \frac{1}{\sqrt{n}} y_1 F_1 + \cdots + \frac{1}{\sqrt{n}} y_d F_d + \frac{1}{n} y_{11} F_{11} + \cdots + \frac{1}{n} y_{dd} F_{dd} \in \mathcal{L} \quad (5.39)$$

It follows that $\mathrm{Tr}(\mathbf{Y}_\epsilon \mathbf{B}) = B_0 y_0 + \frac{1}{\sqrt{n}} \sum_\alpha B_\alpha y_\alpha + \frac{1}{n} \sum_\alpha B_{\alpha\alpha} y_{\alpha\alpha}$ Indeed, $\tilde{C}_n = (\tilde{C}_n^*)^*$. From the bipolar theorem, it follows that $\mathrm{Tr}(\mathbf{Y}_\epsilon \mathbf{B}) \geq 0$ for all $\mathbf{B} \in \tilde{G}_{n,d,\epsilon}^*$ iff $\mathbf{Y}_\epsilon \in \tilde{G}_{n,d,\epsilon}$. Noting that $\mathbf{Y}_\epsilon \in \mathcal{L}$ and following Eq 5.37, it follows that $\mathbf{Y}_\epsilon \in \tilde{G}_{n,d,\epsilon}$ iff $\mathbf{Y}_\epsilon \in \mathcal{V}_+ \oplus (\mathcal{L}^* \cap \mathcal{U}^*)$. Thus, the following theorem on necessary and sufficient conditions follows:

Theorem 5.2.1 *A vector* $(y_0, y_1, \cdots, y_d, y_{11}, y_{22}, \cdots, y_{dd}) \in C_{n,d}$ *if*

$$\mathbf{Y}_{\epsilon=1/(n-1)} \in \mathcal{V}_+ \oplus (\mathcal{L}^* \cap \mathcal{U}^*)$$

and only if

$$\mathbf{Y}_{\epsilon=0} \in \mathcal{V}_+ \oplus (\mathcal{L}^* \cap \mathcal{U}^*)$$

The above two conditions are both expressed as LMIs and can be checked using semi-definite programming. Note that as $n \to \infty$, the necessary and sufficient conditions converge. The above results generalize the results in [12, 13].

REFERENCES

[1] G. Blekherman and J. Lasserre, "The truncated k-moment problem for closure of open sets," *Journal of Functional Analysis*, vol. 263, no. 11, pp. 3604 –3616, 2012.

[2] K. Schmüdgen, *The Moment Problem*. Springer, 2017.

[3] G. Blekherman and L. Fialkow, "The core variety and representing measures in the truncated moment problem," *ArXiv e-prints*, Apr. 2018. arXiv: `1804.04276 [math.FA]`.

[4] A. Einstein, B. Podolsky, and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?" *Phys. Rev.*, vol. 47, pp. 777–780, 10 1935.

[5] J. S. Bell, "On the Einstein Podolsky Rosen paradox," *Physics Physique Fizika*, vol. 1, pp. 195–200, 3 1964.

[6] A. Aspect, P. Grangier, and G. Roger, "Experimental realization of Einstein-Podolsky-Rosen-Bohm Gedanken experiment: A new violation of Bell's inequalities," *Phys. Rev. Lett.*, vol. 49, pp. 91–94, 2 1982.

[7] M. Horodecki, P. Horodecki, and R. Horodecki, "Separability of mixed states: Necessary and sufficient conditions," *Physics Letters A*, vol. 223, no. 1, pp. 1 –8, 1996.

[8] V. Timofte, "On the positivity of symmetric polynomial functions.: Part i: General results," *Journal of Mathematical Analysis and Applications*, vol. 284, no. 1, pp. 174 –190, 2003.

[9] C. Riener, "On the degree and half-degree principle for symmetric polynomials," *Journal of Pure and Applied Algebra*, vol. 216, no. 4, pp. 850 –856, 2012.

[10] P. Görlach, C. Riener, and T. Weiber, "Deciding positivity of multisymmetric polynomials," *Journal of Symbolic Computation*, vol. 74, pp. 603 –616, 2016.

[11] C. Riener, "Symmetric semi-algebraic sets and non-negativity of symmetric polynomials," *Journal of Pure and Applied Algebra*, vol. 220, no. 8, pp. 2809 –2815, 2016.

[12] G. Vitagliano, I. Apellaniz, M. Kleinmann, B. Lcke, C. Klempt, and G. Tóth, "Entanglement and extreme spin squeezing of unpolarized states," *New Journal of Physics*, vol. 19, no. 1, p. 013 027, 2017.

[13] L.-M. Duan, "Entanglement detection in the vicinity of arbitrary dicke states," *Phys. Rev. Lett.*, vol. 107, p. 180 502, 18 2011.